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About the Author

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Dr. Abraham Albert Ungar is a professor of mathematics at North Dakota State University. He received his B.Sc and M.Sc. from the Hebrew University in Jerusalem, and his Ph.D. from Tel-Aviv University.

His areas of interest include mathematical physics, ordinary and partial differential equations, and integral transforms. His interest in the theory of the Laplace transform led him to the development of the method of the *Differential Transform* which is a method of Laplace transform inversion by inspection for solving some linear boundary value problems of interest in mathematical geophysics.

When the author was a young student at the Hebrew University in Jerusalem in the 1960s he was fascinated by the analogies that the $(1+1)$ -dimensional Lorentz transformation group shares with the field of the complex numbers. These analogies suggested to the author that it should be the Lorentz group in higher dimensions that dictates the right way of extending the complex numbers. The desire to extend the field of the complex numbers thus led the author to search for the algebraic structure that regulates the Lorentz group in space dimensions higher than one. He realized that in two or more space dimensions a new effect comes into play, the Thomas precession. Accordingly, it became clear to the author that the right way of extending the complex numbers must involve the abstract Thomas precession, called the Thomas gyration. The Thomas gyration, in turn, suggested the gyro-language that the author uses in this book.

The author's dream from the 1960s to extend the complex numbers by searching for clues hidden in the higher dimensional Lorentz group thus prompted him to decipher the algebraic structure encoded in the Einstein relativistic velocity addition law, discovering that it is regulated by the Thomas precession. The subsequent discovery of the mathematical regularity that the Thomas precession stores and its extension by abstraction resulted in this book which, presenting the theory of gyrogroups and gyrovector spaces, goes beyond Einstein's addition law and its gyroscopic Thomas precession.

