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The late **Carl B. Boyer** was a professor of Mathematics at Brooklyn College and the author of several classic works on the history of Mathematics. **Uta C. Merzbach** earned her PhD in Mathematics and the History of Science from Harvard University, is Curator Emeritus of Mathematics at the Smithsonian Institution, and is the current Director of the LHM Institute.

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