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$B(x_0, \delta)$	Ball centered at $x_0$ with radius $\delta$
$B(\delta)$	Ball centered at origin with radius $\delta$
$\Delta$	The difference operator
$C$	Moment functional
$K(a_n/b_n)$	Continued fraction
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of nonnegative real numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}^+$	The set of nonnegative integers
$\mathbb{C}$	The set of complex numbers
$\Gamma$	The gamma function
$F(a, b, c, z)$	The hypergeometric function
$(x)_n$	The Pochhammer symbol
$P_n^{(\alpha, \beta)}(x)$	Jacobi polynomials
$P_n(x)$	Legendre polynomials
$P_n^*(x)$	Gegenbauer polynomials
$L_n^*(x)$	Laguerre polynomials
$H_n(x)$	Hermite polynomials
$O(x)$	The orbit of $x$
$\Delta^n$	$\Delta^n \Delta^{-1}(\Delta)$
$\prod_{k=0}^{n-1}$	Product
$Sf$	The Schwarzian derivative of $f$
$S$	Shift operator
$f^n$	The $n$ th iterate of $f$
$x^{(n)}$	Factorial polynomial
$\Delta^{-1}$	The antidifference operator