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The basic question each author should pose him/herself, preferably in the future tense before starting, is

Why have we written this book?

In our case the motivation came from many discussions we had with mathematicians, economists, engineers and physicists, mainly working in insurance companies, banks or other financial institutions. Often, these people had as students learnt the more classical theory of stochastics (probability theory, stochastic processes and statistics) and were interested in its applications to insurance and finance. In these discussions notions like *extremes*, *Pareto*, *different moments*, *leptokurtosis*, *tail events*, *Hill estimator* and many, many more would appear. Invariably, a question would follow, "Where can I read more on this?" An answer would usually involve a relatively long list of books and papers with instructions like "For this, look here, for that, perhaps you may find those papers useful, concerning the other, why not read ...". You see the point! After years of frustration concerning the non-existence of a relevant text we decided to write one ourselves. You now hold the fruit of our effort: a book on the modelling of extremal events with special emphasis on applications to insurance and finance. The latter fields of application were mainly motivated by our joint research and teaching at the ETH where various chapters have been used for many years as *Capita Selecta* in the ETH programme on insurance mathematics. Parts of the book have also formed the basis for a Summer School of the Swiss Society of Actuaries (1994) and the Master's Programme in Insurance and Finance at ESSEC, Paris (1995). These trials have invariably led to an increase in the size of the book, due to