CONTENTS

Preface ix

1 Topics in Linear Algebra 1

1.1 Review of Basic Linear Algebra 2
1.2 Linear Independence 7
1.3 The Rank of a Matrix 11
1.4 Main Results on Linear Systems 14
1.5 Eigenvalues 19
1.6 Diagonalization 25
1.7 Quadratic Forms 28
1.8 Quadratic Forms with Linear Constraints 35
1.9 Partitioned Matrices and Their Inverses 37

2 Multivariable Calculus 43

2.1 Gradients and Directional Derivatives 44
2.2 Convex Sets 50
2.3 Concave and Convex Functions I 53
2.4 Concave and Convex Functions II 63
2.5 Quasiconcave and Quasiconvex Functions 68
2.6 Taylor's Formula 77
2.7 Implicit and Inverse Function Theorems 80
2.8 Degrees of Freedom and Functional Dependence 89
2.9 Differentiability 93
2.10 Existence and Uniqueness of Solutions of Systems of Equations 98

3 Static Optimization 103

3.1 Extreme Points 104
3.2 Local Extreme Points 110
3.3 Equality Constraints: The Lagrange Problem 115
3.4 Local Second-Order Conditions 125
3.5 Inequality Constraints: Nonlinear Programming 129
3.6 Sufficient Conditions 135
3.7 Comparative Statics 139
3.8 Nonnegativity Constraints 143
3.9 Concave Programming 148
3.10 Precise Comparative Statics Results 150
3.11 Existence of Lagrange Multipliers 153

4 Topics in Integration 157

4.1 Review of One-Variable Integration 157
4.2 Leibniz's Formula 159
4.3 The Gamma Function 164
4.4 Multiple Integrals over Product Domains 166
4.5 Double Integrals over General Domains 171
4.6 The Multiple Riemann Integral 175
4.7 Change of Variables 178
4.8 Generalized Double Integrals 186

5 Differential Equations I: First-Order Equations in One Variable 189

5.1 Introduction 190
5.2 The Direction is Given: Find the Path! 193
5.3 Separable Equations 194
5.4 First-Order Linear Equations 200
5.5 Exact Equations and Integrating Factors 206
5.6 Transformation of Variables 208
5.7 Qualitative Theory and Stability 211
5.8 Existence and Uniqueness 217

6 Differential Equations II: Second-Order Equations and Systems in the Plane 223

6.1 Introduction 223
6.2 Linear Differential Equations 226
6.3 Constant Coefficients 228
6.4 Stability for Linear Equations 235
6.5 Simultaneous Equations in the Plane 237
6.6 Equilibrium Points for Linear Systems 243
6.7 Phase Plane Analysis 246
6.8 Stability for Nonlinear Systems 251
6.9 Saddle Points 255

7 Differential Equations III: Higher-Order Equations 259

7.1 Linear Differential Equations 259
7.2 The Constant Coefficients Case 263
7.3 Stability of Linear Differential Equations 266
7.4 Systems of Differential Equations 269
7.5 Stability for Nonlinear Systems 273
7.6 Qualitative Theory 278
7.7 A Glimpse at Partial Differential Equations 280

8 Calculus of Variations 287

8.1 The Simplest Problem 288
8.2 The Euler Equation 290
8.3 Why the Euler Equation is Necessary 293
8.4 Optimal Savings 298
8.5 More General Terminal Conditions 300

9 Control Theory: Basic Techniques 305

9.1 The Basic Problem 306
9.2 A Simple Case 308
9.3 Regularity Conditions 312
9.4 The Standard Problem 314
9.5 The Maximum Principle and the Calculus of Variations 322
9.6 Adjoint Variables as Shadow Prices 324
9.7 Sufficient Conditions 330
9.8 Variable Final Time 336
9.9 Current Value Formulations 338
9.10 Scrap Values 341
9.11 Infinite Horizon 348
9.12 Phase Diagrams 353

10 Control Theory with Many Variables 359

10.1 Several Control and State Variables 360
10.2 Some Examples 366
10.3 Infinite Horizon 370
10.4 Existence Theorems and Sensitivity 373
10.5 A Heuristic Proof of the Maximum Principle 377
10.6 Mixed Constraints 380
10.7 Pure State Constraints 383
10.8 Generalizations 386

11 Difference Equations 389

11.1 First-Order Difference Equations 390
11.2 Economic Applications 396
11.3 Second-Order Difference Equations 401
11.4 Linear Equations with Constant Coefficients 404
11.5 Higher-Order Equations 410
11.6 Systems of Difference Equations 415
11.7 Stability of Nonlinear Difference Equations 419

12 Discrete Time Optimization 423

12.1 Dynamic Programming 424
12.2 The Euler Equation 433
12.3 Infinite Horizon 435

12.4 The Maximum Principle 441
12.5 More Variables 444
12.6 Stochastic Optimization 448
12.7 Infinite Horizon Stationary Problems 458

13 Topology and Separation 465

13.1 Point Set Topology in $\mathbb{R}^n$ 465
13.2 Topology and Convergence 471
13.3 Continuous Functions 475
13.4 Maximum Theorems 481
13.5 Convex Sets 486
13.6 Separation Theorems 491
13.7 Productive Economies and Frobenius's Theorem 495

14 Correspondences and Fixed Points 499

14.1 Correspondences 500
14.2 A General Maximum Theorem 509
14.3 Fixed Points for Contraction Mappings 513
14.4 Brouwer's and Kakutani's Fixed Point Theorems 516
14.5 Equilibrium in a Pure Exchange Economy 521

Appendix A Sets, Completeness, and Convergence 525

A.1 Sets and Functions 525
A.2 Least Upper Bound Principle 530
A.3 Sequences of Real Numbers 533
A.4 Infimum and Supremum of Functions 541

Appendix B Trigonometric Functions 545

B.1 Basic Definitions and Results 545
B.2 Differentiating Trigonometric Functions 551
B.3 Complex Numbers 555

Answers 559

References 605

Index 609