Contents

1 Events and their probabilities

1.1 Introduction 1
1.2 Events as sets 1
1.3 Probability 4
1.4 Conditional probability 8
1.5 Independence 13
1.6 Completeness and product spaces 14
1.7 Worked examples 16
1.8 Problems 21

2 Random variables and their distributions

2.1 Random variables 26
2.2 The law of averages 30
2.3 Discrete and continuous variables 33
2.4 Worked examples 35
2.5 Random vectors 38
2.6 Monte Carlo simulation 41
2.7 Problems 43

3 Discrete random variables

3.1 Probability mass functions 46
3.2 Independence 48
3.3 Expectation 50
3.4 Indicators and matching 56
3.5 Examples of discrete variables 60
3.6 Dependence 62
3.7 Conditional distributions and conditional expectation 67
3.8 Sums of random variables 70
3.9 Simple random walk 71
3.10 Random walk: counting sample paths 75
3.11 Problems 83

4 Continuous random variables

4.1 Probability density functions 89
4.2 Independence 91
4.3 Expectation 93
4.4 Examples of continuous variables 95
4.5 Dependence 98
4.6 Conditional distributions and conditional expectation 104
4.7 Functions of random variables 107
4.8 Sums of random variables 113
4.9 Multivariate normal distribution 115
4.10 Distributions arising from the normal distribution 119
4.11 Sampling from a distribution 122
4.12 Coupling and Poisson approximation 127
4.13 Geometrical probability 133
4.14 Problems 140

5 Generating functions and their applications

5.1 Generating functions 148
5.2 Some applications 156
5.3 Random walk 162
5.4 Branching processes 171
5.5 Age-dependent branching processes 175
5.6 Expectation revisited 178
5.7 Characteristic functions 181
5.8 Examples of characteristic functions 186
5.9 Inversion and continuity theorems 189
5.10 Two limit theorems 193
5.11 Large deviations 201
5.12 Problems 206

6 Markov chains

6.1 Markov processes 213
6.2 Classification of states 220
6.3 Classification of chains 223
6.4 Stationary distributions and the limit theorem 227
6.5 Reversibility 237
6.6 Chains with finitely many states 240
6.7 Branching processes revisited 243
6.8 Birth processes and the Poisson process 246
6.9 Continuous-time Markov chains 256
6.10 Uniform semigroups 266
6.11 Birth–death processes and imbedding 268
6.12 Special processes 274
6.13 Spatial Poisson processes 281
6.14 Markov chain Monte Carlo 291

7 Convergence of random variables

7.1 Introduction 305
7.2 Modes of convergence 308
7.3 Some ancillary results 318
7.4 Laws of large numbers 325
7.5 The strong law 329
7.6 The law of the iterated logarithm 332
7.7 Martingales 333
7.8 Martingale convergence theorem 338
7.9 Prediction and conditional expectation 343
7.10 Uniform integrability 350
7.11 Problems 354

8 Random processes

8.1 Introduction 360
8.2 Stationary processes 361
8.3 Renewal processes 365
8.4 Queues 367
8.5 The Wiener process 370
8.6 Existence of processes 371
8.7 Problems 373

9 Stationary processes

9.1 Introduction 375
9.2 Linear prediction 377
9.3 Autocovariances and spectra 380
9.4 Stochastic integration and the spectral representation 387
9.5 The ergodic theorem 393
9.6 Gaussian processes 405
9.7 Problems 409

10 Renewals

10.1 The renewal equation 412
10.2 Limit theorems 417
10.3 Excess life 421
10.4 Applications 423
10.5 Renewal–reward processes 431
10.6 Problems 437

11 Queues

11.1 Single-server queues 440
11.2 M/M/1 442
11.3 M/G/1 445
11.4 G/M/1 451
11.5 G/G/1 455

11.6 Heavy traffic 462
11.7 Networks of queues 462
11.8 Problems 468

12 Martingales

12.1 Introduction 471
12.2 Martingale differences and Hoeffding's inequality 476
12.3 Crossings and convergence 481
12.4 Stopping times 487
12.5 Optional stopping 491
12.6 The maximal inequality 496
12.7 Backward martingales and continuous-time martingales 499
12.8 Some examples 503
12.9 Problems 508

13 Diffusion processes

13.1 Introduction 513
13.2 Brownian motion 514
13.3 Diffusion processes 516
13.4 First passage times 525
13.5 Barriers 530
13.6 Excursions and the Brownian bridge 534
13.7 Stochastic calculus 537
13.8 The Itô integral 539
13.9 Itô's formula 544
13.10 Option pricing 547
13.11 Passage probabilities and potentials 554
13.12 Problems 561

Appendix I. Foundations and notation 564

Appendix II. Further reading 569

Appendix III. History and varieties of probability 571

Appendix IV. John Arbuthnot's Preface to *Of the laws of chance* (1692) 573

Appendix V. Table of distributions 576

Appendix VI. Chronology 578

Bibliography 580

Notation 583

Index 585