Contents

1 Review of Aggregation Functions 1
1.1 Aggregation Functions 1
1.2 Applications of Aggregation Functions 5
1.3 Classification and General Properties 9
1.3.1 Main Classes 9
1.3.2 Main Properties 10
1.3.3 Duality 18
1.3.4 Comparability 21
1.3.5 Continuity and Stability 21
1.4 Main Families and Prototypical Examples 24
1.4.1 Min and Max 24
1.4.2 Means 24
1.4.3 Medians 25
1.4.4 Ordered Weighted Averaging 26
1.4.5 Choquet and Sugeno Integrals 26
1.4.6 Conjunctive and Disjunctive Functions 27
1.4.7 Mixed Aggregation 28
1.5 Composition and Transformation of Aggregation Functions 29
1.6 How to Choose an Aggregation Function 32
1.7 Supplementary Material: Some Methods for Approximation and Optimization 35
1.7.1 Univariate Approximation and Smoothing 35
1.7.2 Approximation with Constraints 37
1.7.3 Multivariate Approximation 38
1.7.4 Convex and Non-convex Optimization 41
1.7.5 Main Tools and Libraries 47
References 50

2 Classical Averaging Functions 55
2.1 Semantics 55
2.1.1 Measure of Orness 56
2.2 Classical Means 58
2.2.1 Arithmetic Mean 58
2.3 Weighted Quasi-arithmetic Means 65
2.3.1 Definitions 65
2.3.2 Main Properties 66
2.3.3 Examples 69
2.3.4 Calculation 73
2.3.5 Weighting Triangles 73
2.3.6 Weights Dispersion 76
2.3.7 How to Choose Weights 77
2.4 Other Means 82
2.4.1 Gini Means 82
2.4.2 Bonferroni Means 83
2.4.3 Heronian Mean 85
2.4.4 Generalized Logarithmic Means 86
2.4.5 Cauchy and Lagrangean Means 88
2.4.6 Mean of Bajraktarevic 89
2.4.7 Mixture Functions 89
2.4.8 Compound Means 90
2.4.9 Extending Bivariate Means to More Than Two Arguments 91
References 97

3 Ordered Weighted Averaging 101
3.1 Definitions 101
3.2 Main Properties 102
3.2.1 Orness Measure 103
3.2.2 Entropy 105
3.3 Other Types of OWA Functions 106
3.3.1 Neat OWA 106
3.3.2 Generalized OWA 107
3.3.3 Weighted OWA 109
3.4 How to Choose Weights in OWA 114
3.4.1 Methods Based on Data 114
3.4.2 Methods Based on a Measure of Dispersion 117
3.4.3 Methods Based on Weight Generating Functions 120
3.4.4 Fitting Weight Generating Functions 123
3.4.5 Choosing Parameters of Generalized OWA 126
3.5 Induced OWA 130
3.5.1 Definition 130
3.5.2 Properties 130

3.5.3 Induced Generalized OWA . . . 133
3.5.4 Choices for the Inducing Variable . . . 134
3.6 Medians and Order Statistics . . . 137
3.6.1 Median . . . 137
3.6.2 Order Statistics . . . 141
References . . . 141

4 Fuzzy Integrals . . . 145
4.1 Choquet Integral . . . 145
4.1.1 Semantics . . . 145
4.1.2 Definitions and Properties . . . 147
4.1.3 Types of Fuzzy Measures . . . 153
4.1.4 Interaction, Importance and Other Indices . . . 160
4.1.5 Special Cases of the Choquet Integral . . . 164
4.1.6 Fitting Fuzzy Measures . . . 166
4.1.7 Generalized Choquet Integral . . . 173
4.2 Sugeno Integral . . . 174
4.2.1 Definition and Properties . . . 174
4.2.2 Special Cases . . . 175
4.3 Induced Fuzzy Integrals . . . 177
References . . . 178

5 Penalty Based Averages . . . 183
5.1 Motivation and Definitions . . . 183
5.2 Types of Penalty Functions . . . 186
5.2.1 Faithful Penalty Functions . . . 186
5.2.2 Restricted Dissimilarity Functions . . . 188
5.2.3 Minkowski Gauge Based Penalties . . . 191
5.3 Examples . . . 193
5.3.1 Quasi-arithmetic Means, OWA and Choquet Integral . . . 193
5.3.2 Deviation Means . . . 194
5.3.3 Entropic Means . . . 195
5.3.4 Bregman Loss Functions . . . 195
5.4 New Penalty Based Aggregation Functions . . . 197
5.5 Relation to the Maximum Likelihood Principle . . . 201
5.6 Representation of Averages . . . 203
References . . . 204

6 More Types of Averaging and Construction Methods . . . 207
6.1 Some Construction Methods . . . 207
6.1.1 Idempotization . . . 207
6.1.2 Means Defined by Using Graduation Curves . . . 208
6.1.3 Aggregation Functions with Flying Parameter . . . 210
6.1.4 Construction of Shift-Invariant Functions . . . 212
6.1.5 Interpolatory Constructions . . . 212

6.2 Other Types of Aggregation and Properties 216
6.2.1 Bi-capacities 216
6.2.2 Linguistic Aggregation Functions 217
6.2.3 Multistage Aggregation 217
6.2.4 Migrativity 218
6.3 Overlap and Grouping Functions 218
6.3.1 Definition of Overlap Functions and Basic Properties 219
6.3.2 Characterization of Overlap Functions 220
6.3.3 Homogeneous Overlap Functions 221
6.3.4 k-Lipschitz Overlap Functions 222
6.3.5 n-dimensional Overlap Functions 224
6.3.6 Grouping Functions 225
6.4 Generalized Bonferroni Mean 227
6.4.1 Main Definitions 227
6.4.2 Properties of the Generalized Bonferroni Mean 229
6.4.3 Replacing the Outer Mean 230
6.4.4 Replacing the Inner Mean 231
6.4.5 Replacing the Product Operation 232
6.4.6 Extensions to $B_{\mathrm{M}}^{\mathrm{K}}$ 233
6.4.7 Boundedness of the Generalized Bonferroni Mean 234
6.4.8 k-intolerance Boundedness 237
6.4.9 Generated t-norm and Generated Quasi-arithmetic Means as Components of $B_{\mathbb{M}}$ 238
6.5 Consistency and Stability 239
6.5.1 Motivation 239
6.5.2 Strictly Stable Families 240
6.5.3 R-strict Stability 241
6.5.4 Learning Consistent Weights 245
6.5.5 Consistency and Global Monotonicity 246
References 247

7 Non-monotone Averages 251
7.1 Motivation 251
7.2 Weakly Monotone Functions 252
7.2.1 Basic Properties of Weakly Monotone Functions 254
7.3 Robust Estimators of Location 256
7.3.1 Mode 257
7.3.2 Shorth 257
7.3.3 Least Median of Squares (LMS) 258
7.3.4 Least Trimmed Squares (LTS) 258
7.3.5 Least Trimmed Absolute Deviations (LTA) 259
7.3.6 The Least Winsorized Squares Estimator 260
7.3.7 OWA Penalty Functions 260

7.4 Lehmer and Gini Means 261
7.4.1 Lehmer Means 261
7.4.2 Gini Means 263
7.5 Mixture Functions 265
7.5.1 Some Special Cases of Weighting Functions 266
7.5.2 Affine Weighting Functions 267
7.5.3 Linear Combinations of Weighting Functions 269
7.5.4 The Duals of Lehmer Mean and Other Mixture Functions 270
7.6 Density Based Means and Medians 271
7.6.1 Density Based Means 271
7.6.2 Density Based Medians and ML Estimators 272
7.6.3 Modified Weighting Functions 273
7.7 Mode-Like Averages 275
7.8 Spatial-Tonal Filters 277
7.9 Transforms 278
7.10 Cone Monotone Functions 281
7.10.1 Formal Definitions and Properties 281
7.10.2 Verification of Cone Monotonicity 285
7.10.3 Construction of Cone Monotone Lipschitz Functions 286
7.11 Monotonicity with Respect to Coalitions 288
7.11.1 Simple Majority 288
7.11.2 Majority and Preferential Inputs 290
7.11.3 Coalitions 291
7.12 Directional Monotonicity 293
7.12.1 Properties of r-Monotone functions 294
7.12.2 The Set of Directions of Increasingness 295
7.13 Pre-aggregation Functions 296
7.13.1 Definitions and Properties 296
7.13.2 Construction of Pre-aggregation Functions by Composition 297
7.13.3 Choquet-Like Construction Method of Pre-aggregation Functions 299
7.13.4 Sugeno-Like Construction Method of Pre-aggregation Functions 300
References 302

8 Averages on Lattices 305
8.1 Aggregation of Intervals and Intuitionistic Fuzzy Values 305
8.1.1 Preliminary Definitions 306
8.1.2 Aggregation on Product Lattices 306
8.1.3 Arithmetic Means and OWA for AIFV 309
8.1.4 Alternative Definitions of Aggregation Functions on AIFV 313

8.1.5 Consistency with Operations on Ordinary Fuzzy Sets 316
8.1.6 Medians for AIFV 319
8.1.7 Bonferroni Means 320
8.2 Medians on Lattices 327
8.2.1 Medians as Penalty Based Functions. 327
8.2.2 Median Graphs and Distributive Lattices. 328
8.2.3 Medians on Infinite Lattices and Fermat Points 330
8.2.4 Medians Based on Distances Between Intervals 331
8.2.5 Numerical Comparison 333
8.3 Penalty Functions on Cartesian Products of Lattices 336
References 342

Index 347