Contents

Introduction 1

1 Basic notions 7
- 1.1 MV-algebras . . . 7
- 1.2 Homomorphisms and ideals . . . 12
- 1.3 Subdirect representation theorem . . . 19
- 1.4 MV-equations . . . 20
- 1.5 Boolean algebras . . . 24
- 1.6 MV-chains . . . 27
- 1.7 Bibliographical remarks . . . 29

2 Chang completeness theorem 31
- 2.1 The functor Γ . . . 31
- 2.2 Good sequences . . . 34
- 2.3 The partially ordered monoid M_A . . . 37
- 2.4 Chang's ℓ-group G_A . . . 40
- 2.5 Chang completeness theorem . . . 43
- 2.6 Bibliographical remarks . . . 49

3 Free MV-algebras 51
- 3.1 McNaughton functions . . . 51
- 3.2 The one-dimensional case . . . 56
- 3.3 Decomposing McNaughton functions . . . 62
- 3.4 Ideals in free MV-algebras . . . 64
- 3.5 Simple MV-algebras . . . 70
- 3.6 Semisimple MV-algebras . . . 72
- 3.7 Bibliographical remarks . . . 75

4 Łukasiewicz ∞-valued calculus **77**
4.1 Many-valued propositional calculi 78
4.2 Wajsberg algebras 82
4.3 Provability 87
4.4 Lindenbaum algebra 92
4.5 All tautologies are provable 94
4.6 Syntactic and semantic consequence 97
4.7 Bibliographical remarks 101

5 Ulam's game **103**
5.1 Questions and answers 103
5.2 Dynamics of states of knowledge 104
5.3 Operations on states of knowledge 107
5.4 Bibliographical remarks 109

6 Lattice-theoretical properties **111**
6.1 Minimal prime ideals 112
6.2 Stonean ideals and archimedean elements 115
6.3 Hyperarchimedean algebras 116
6.4 Direct products 121
6.5 Boolean products of MV-algebras 124
6.6 Completeness 129
6.7 Atoms and Pseudocomplements 132
6.8 Complete distributivity 134
6.9 Bibliographical remarks 137

7 MV-algebras and ℓ-groups **139**
7.1 Inverting the functor Γ 139
7.2 Applications 146
7.3 The radical 150
7.4 Perfect MV-algebras 151
7.5 Bibliographical remarks 156

8 Varieties of MV-algebras **157**
8.1 Basic definitions 157
8.2 Varieties from simple algebras 160
8.3 MV-chains of finite rank 161

8.4 Komori's classification 167
8.5 Varieties generated by a finite chain 171
8.6 The cardinality of $Free_r^n$ 173
8.7 Bibliographical remarks 177

9 Advanced topics **179**
9.1 McNaughton's theorem 180
9.2 Nonsingular fans and normal forms 185
9.3 Complexity of the tautology problem 187
9.4 MV-algebras and AF C*-algebras 191
9.5 Di Nola's representation theorem 193
9.6 Bibliographical remarks 194

10 Further Readings **197**
10.1 More than two truth values 197
10.2 Current Research Topics 199
10.2.1 Product 199
10.2.2 States, Observables, Probability, Partitions 200
10.2.3 Deduction 201
10.2.4 Further constructions 201

Bibliography **203**

Index **225**