Contents

1 Introduction 1

1.1 Motivation for the Research . . . 1

1.2 Structure of the Book . . . 2

2 Literature 5

2.1 Knowledge Network Metaphor . . . 5

2.1.1 Structure . . . 6

2.1.2 Understanding . . . 7

2.1.3 Mechanical Understanding . . . 9

2.2 Concept Development Theories . . . 11

2.2.1 Procedural and Conceptual Knowledge in Mathematics . . . 11

2.2.2 APOS Theory . . . 12

2.2.3 Procept Theory . . . 13

2.2.4 Operational and Structural Thinking . . . 14

2.2.5 Theory of Abstraction in Context . . . 15

2.2.6 Theory of Isolated and Generic Models . . . 17

2.3 Advanced Mathematical Thinking . . . 20

2.3.1 Generalisation and Abstraction . . . 21

2.3.2 Proving . . . 22

2.3.3 Natural and Formal Learners . . . 23

2.3.4 Concept Image and Concept Definition . . . 24

2.4 Constructivist Approaches to Mathematics Teaching . . . 26

2.5 Problem Solving . . . 27

2.6 Summary . . . 28

3 General Overview of Methodology 31

3.1 Introduction . . . 31

3.2 Purposes of Research . . . 32

3.3 Research Tool . . . 33
3.3.1 Research Question . . . 34
3.3.2 Data Collection – Clinical Interviews . . . 34
3.4 Summary . . . 36

4 Mathematical Background of the Research **37**
4.1 Introduction . . . 37
4.2 Description of RA in Mathematical Terms . . . 38
4.2.1 Preliminaries . . . 38
4.2.2 Main Part . . . 39
4.3 Description of RA in Didactic Terms . . . 44
4.3.1 Reduction . . . 44
4.3.2 Operations of z-addition and z-multiplication . . . 46
4.4 Selected Problems . . . 46
4.4.1 Reduction and Inverse Reduction . . . 47
4.4.2 Additive Equations $a \oplus x = b$ . . . 48
4.4.3 Multiplicative Equations $c \otimes x = d$ . . . 51
4.4.4 On z-divisibility Tests . . . 54
4.4.5 On z-squares and z-square Roots . . . 55
4.4.6 Quadratic Equations . . . 58
4.4.7 Higher Powers and Sequences . . . 59
4.4.8 Abstract Algebra . . . 60
4.4.9 Recreational Mathematics . . . 61
4.5 Restricted Arithmetic and Abstract Algebra . . . 61
4.6 Conclusions . . . 63

5 Methodology – Pilot Study **65**
5.1 Pre-Pilot Study – Introspection . . . 65
5.2 Pilot Study . . . 66
5.2.1 Interview 1 . . . 67
5.2.2 Interview 2 . . . 70
5.2.3 Interview 3 . . . 71
5.3 Pragmatics of Data Analysis of the Pilot Study . . . 75
5.4 Students' Comprehension of Structure – First Attempt . . . 76
5.5 Conclusions for the Pilot Study . . . 78
5.6 Implications for the Main Study . . . 78

6 Methodology – Main Study **81**
6.1 Research Questions 81
6.2 Data Collection – Clinical Interviews 81
6.3 Participants 82
6.4 Data Analysis of the Main Study 84
6.4.1 Grounded Theory – Theoretical Overview 84
6.4.2 Atomic Analysis 90
6.5 Pragmatics of Data Analysis 93
6.5.1 Open Coding – Pathways and Categories 94
6.5.2 Axial Coding – Structure of Categories 100
6.5.3 Central Category 102

7 Building Networks **103**
7.1 Introduction 103
7.2 Note on Theories Used for Interpretation 103
7.3 Abbreviations Used in the Chapter 104
7.4 Epistemic Actions in Building Networks 105
7.5 Interpretation of z-numbers 106
7.6 Concept Development 110
7.6.1 Reduction and its Inverse 110
7.6.2 Equations $a \oplus x = b$, $c \otimes x = d$ 115
7.6.3 Zero Divisors 120
7.6.4 Operation of z-subtraction and z-negation 122
7.6.5 Operation of z-division 126
7.7 Abstract Algebra Concepts 130
7.8 Learning Styles 134
7.9 Analogy Versus New Structure – Theory of Comprehending Structure 135
7.9.1 Analogies as Obstacles 139
7.9.2 Regularities and Anomalies 141
7.10 U-Shape Development 143
7.11 Conclusions for the Chapter 144

8 Case Study – Longitudinal Study **147**
8.1 Introduction 147
8.2 Methods 148
8.2.1 Purposes and Research Questions 148
8.2.2 Participant 148

8.2.3 Description of The Interaction . . . 149
8.2.4 Data Collection – Case Study Database . . . 152
8.3 Pragmatics of Data analysis . . . 152
8.3.1 Note on Roles . . . 153
8.3.2 Analysing Situations of The Interaction . . . 154
8.3.3 Motivation . . . 159
8.4 Small Scale Conceptions . . . 160
8.4.1 Understanding of Inverse Reduction . . . 161
8.4.2 Operation of z-subtraction . . . 164
8.4.3 Operation of z-division . . . 167
8.4.4 Zero Divisors . . . 169
8.5 Global Conception . . . 172
8.5.1 Ownership . . . 172
8.5.2 Pythagorean Triples . . . 174
8.5.3 On z-squares and Higher Powers . . . 177
8.5.4 Mathematical Writing . . . 181
8.6 Conclusions for the Chapter . . . 186

9 Findings and Conclusions **191**
9.1 Comprehension of Structure . . . 191
9.2 Evaluation of Research Tools . . . 194
9.2.1 Didactic Potential of RA and Other Mathematical Topics . . . 194
9.2.2 Constructivist Approach . . . 196
9.2.3 Methodological Results . . . 198
9.3 Pedagogical Implications and Further Research . . . 198

10 References **203**

Index **211**

Appendices **219**

A Mathematical Project on Restricted Arithmetic **219**

B Some Tables **221**

C List of Groups in RA **223**
C.1 Additive Groups . . . 223

C.2 Multiplicative Groups 224

D Survey of Experiments **227**
D.1 Anita (Group A) 227
D.2 Betty (Group A) 228
D.3 Dana (Group C) 228
D.4 Frank (Group B) 229
D.5 Gill (Group C) 229
D.6 John (Group B) 229
D.7 Kathy (Group B) 229
D.8 Ruth (Group A) 230
D.9 Susan (Group B) 230
D.10 Tony (Group B) 230
D.11 Vera (Group C) 230
D.12 Molly (Group A) 231

E Content of Molly's Writing **235**