Contents

Preface ix

Chapter 1. Introduction 1

§1.1. The brachistochrone 1

§1.2. The terrestrial brachistochrone 5

§1.3. Geodesics 7

§1.4. Minimal surfaces 14

§1.5. Recommended reading 18

§1.6. Exercises 20

Chapter 2. The First Variation 27

§2.1. The simplest problem 27

§2.2. Euler's approach 28

§2.3. Lagrange's approach 32

§2.4. Recommended reading 45

§2.5. Exercises 46

Chapter 3. Cases and Examples 47

§3.1. Special cases 47

§3.2. Case study: Minimal surface of revolution 52

§3.3. Case study: The brachistochrone 59

§3.4. Geodesics 62
§3.5. Recommended reading 63
§3.6. Exercises 64

Chapter 4. Basic Generalizations 69
§4.1. Higher-order derivatives 69
§4.2. Case study: The cantilever beam 71
§4.3. Multiple unknown functions 77
§4.4. Lagrangian mechanics 79
§4.5. Case study: The spherical pendulum 85
§4.6. Hamiltonian mechanics 88
§4.7. Ray optics 93
§4.8. Double integrals 96
§4.9. Recommended reading 101
§4.10. Exercises 101

Chapter 5. Constraints 107
§5.1. Types of constraints 107
§5.2. Lagrange multipliers 114
§5.3. Isoperimetric constraints 117
§5.4. Case study: Queen Dido's problem 120
§5.5. Case study: Euler's elastica 123
§5.6. Holonomic constraints 129
§5.7. Case study: A sliding rod 133
§5.8. Recommended reading 136
§5.9. Exercises 136

Chapter 6. The Second Variation 139
§6.1. Introduction 139
§6.2. Legendre's condition 142
§6.3. Jacobi's condition 147
§6.4. Case study: The catenoid revisited 160

§6.5. Recommended reading 167
§6.6. Exercises 168

Chapter 7. Review and Preview 171
§7.1. Introduction 171
§7.2. Necessary conditions 172
§7.3. Sufficient conditions 173
§7.4. Two dependent variables 173
§7.5. History and preview 175
§7.6. Recommended reading 176

Chapter 8. The Homogeneous Problem 177
§8.1. Integrals in parametric form 177
§8.2. Euler–Lagrange equations 180
§8.3. The Weierstrass equation 182
§8.4. Case study: The parametric Queen Dido problem 186
§8.5. Recommended reading 189
§8.6. Exercises 190

Chapter 9. Variable-Endpoint Conditions 191
§9.1. Natural boundary conditions 191
§9.2. Transversality conditions 197
§9.3. Focal points 206
§9.4. Case study: Neile's parabola 206
§9.5. Recommended reading 211
§9.6. Exercises 212

Chapter 10. Broken Extremals 215
§10.1. The Weierstrass–Erdmann corner conditions 215
§10.2. Carathéodory's indicatrix 224
§10.3. Recommended reading 231
§10.4. Exercises 232

Chapter 11. Strong Variations 235
§11.1. Troubles with weak variations 235
§11.2. Weierstrass's condition 242
§11.3. Case study: Newton's problem 247
§11.4. Recommended reading 254
§11.5. Exercises 254
Chapter 12. Sufficient Conditions 255
§12.1. Introduction 255
§12.2. Fields of extremals 256
§12.3. Hilbert's invariant integral 260
§12.4. Weierstrass's E-function revisited 262
§12.5. The royal road 267
§12.6. Recommended reading 273
§12.7. Exercises 274
Bibliography 277
Index 295