Contents

Dedication x

1.11.2

	Preface xi
	Symbols and Abbreviations xiii
1	Mathematical Foundations 1: Point-Set Concepts, Set and Measur Functions, Normed Linear Spaces, and Integration 1
1.1	Set Notation and Operations 1
1.1.1	Sets and Set Inclusion 1
1.1.2	Set Algebra 2
1.2	Single-Valued Functions 4
1.3	Real and Extended Real Numbers 6
1.4	Metric Spaces 7
1.5	Limits of Sequences 8
1.6	Point-Set Theory 10
1.7	Continuous Functions 12
1.8	Operations on Sequences of Sets 13
1.9	Classes of Subsets of Ω 15
1.9.1	Topological Space 15
1.9.2	σ -Algebra of Sets and the Borel σ -Algebra 15
1.10	Set and Measure Functions 17
1.10.1	Set Functions 17
1.10.2	Measure Functions 18
1.10.3	Outer Measure Functions 19
1.10.4	Complete Measure Functions 21
1.10.5	Lebesgue Measure 21
1.10.6	Measurable Functions 23
1.10.7	Lebesgue Measurable Functions 26
1.11	Normed Linear Spaces 27
1.11.1	Space of Bounded Real-Valued Functions 27

Space of Bounded Continuous Real-Valued Functions 28

1.11.3	Some Classical Banach Spaces 29
1.12	Integration 31
1.12.1	Integral of a Non-negative Simple Function 32
1.12.2	Integral of a Non-negative Measurable Function Using Simple
	Functions 33
1.12.3	Integral of a Measurable Function 33
1.12.4	Integral of a Measurable Function on a Measurable Set 34
1.12.5	Convergence of Sequences of Functions 35
2	Mathematical Foundations 2: Probability, Random Variables, and
	Convergence of Random Variables 37
2.1	Probability Spaces 37
2.2	Probability Distributions 42
2.3	The Expectation of a Random Variable 49
2.3.1	Theoretical Underpinnings 49
2.3.2	Computational Considerations 50
2.4	Moments of a Random Variable 52
2.5	Multiple Random Variables 54
2.5.1	The Discrete Case 54
2.5.2	The Continuous Case 59
2.5.3	Expectations and Moments 63
2.5.4	The Multivariate Discrete and Continuous Cases 69
2.6	Convergence of Sequences of Random Variables 72
2.6.1	Almost Sure Convergence 73
2.6.2	Convergence in $L^p, p > 0$ 73
2.6.3	Convergence in Probability 75
2.6.4	Convergence in Distribution 75
2.6.5	Convergence of Expectations 76
2.6.6	Convergence of Sequences of Events 78
2.6.7	Applications of Convergence of Random Variables 79
2.7	A Couple of Important Inequalities 80
Append	ix 2.A The Conditional Expectation $E(X Y)$ 81
3	Mathematical Foundations 3: Stochastic Processes, Martingales, and
	Brownian Motion 85
3.1	Stochastic Processes 85
3.1.1	Finite-Dimensional Distributions of a Stochastic Process 86
3.1.2	Selected Characteristics of Stochastic Processes 88
3.1.3	Filtrations of A 89 30 anothern's aldernamin suggested 7.0
3.2	Martingales 91
3.2.1	Discrete-Time Martingales 91
3.2.1.1	Discrete-Time Martingale Convergence 93

Continuous-Time Martingales

3.2.2

3.2.2.1	Contin	nuous-1 ime Martingale Convergence 9/				
3.2.3	Martingale Inequalities 97					
3.3	Path Regularity of Stochastic Processes 98					
3.4	Symmetric Random Walk 99					
3.5	Brownian Motion 100					
3.5.1	Standa	ard Brownian Motion 100				
3.5.2	BM as a Markov Process 104					
3.5.3	Constructing BM 106					
3.5.3.1	BM C	onstructed from N(0, 1) Random Variables 106				
3.5.3.2		the Limit of Symmetric Random Walks 108				
3.5.4		Noise Process 109				
Append		Kolmogorov Existence Theorem: Another Look 109				
	dix 3.B	Nondifferentiability of BM 110				
4		matical Foundations 4: Stochastic Integrals, Itô's Integral, Itô	ô's			
	Formu	la, and Martingale Representation 113 motorborns				
4.1		uction 113 TZ antismizorga A not asmed 52 syllarett				
4.2						
4.3						
4.4						
4.5						
Append						
••						
5	Stocha	astic Differential Equations 133 M DIANO-HORE SATE				
5.1	Introduction 133 I would be some greened to start and the					
5.2	Existence and Uniqueness of Solutions 134					
5.3	Linear SDEs 136 management marking bus M3 ad T 4 Cyclonogo					
5.3.1	Strong Solutions to Linear SDEs 137					
5.3.2	_	rties of Solutions 147 815 emerge distributed in				
5.3.3	-	ons to SDEs as Markov Processes 152				
5.4		and Stability 154				
Append		Solutions of Linear SDEs in Product Form (Evans, 2013;				
5.A.1	Linear					
5.A.2		Variety 161				
		Integrating Factors and Variation of Parameters 162				
5.B.1	Integrating Factors 163					
5.B.2		ion of Parameters 164 ORS SHIPPON DESCO SATE CALL				
6	Stocha	astic Population Growth Models 167				
6.1		uction 167 and williams Probability Dist/761 on out				
6.2		erministic Population Growth Model 168				

0.5	A Stochastic Population Growth Model 109
6.4	Deterministic and Stochastic Logistic Growth Models 170
6.5	Deterministic and Stochastic Generalized Logistic Growth Models 174
6.6	Deterministic and Stochastic Gompertz Growth Models 177
6.7	Deterministic and Stochastic Negative Exponential Growth
124	Models 179
6.8	Deterministic and Stochastic Linear Growth Models 181
6.9	Stochastic Square-Root Growth Model with Mean Reversion 182
	ix 6.A Deterministic and Stochastic Logistic Growth Models with a Allee Effect 184
Append	ix 6.B Reducible SDEs 189
3	
7	Approximation and Estimation of Solutions to Stochastic
	Differential Equations 193
7.1	Introduction 193
7.2	Iterative Schemes for Approximating SDEs 194
7.2.1	The EM Approximation 194
7.2.2	Strong and Weak Convergence of the EM Scheme 196
7.2.3	The Milstein (Second-Order) Approximation 196
7.3	The Lamperti Transformation 199
7.4	Variations on the EM and Milstein Schemes 203
7.5	Local Linearization Techniques 205
7.5.1	The Ozaki Method 205
7.5.2	The Shoji–Ozaki Method 207
7.5.3	The Rate of Convergence of the Local Linearization Method 211
Append	ix 7.A Stochastic Taylor Expansions 212
Append	ix 7.B The EM and Milstein Discretizations 217
7.B.1	The EM Scheme 217
7.B.2	The Milstein Scheme 218
Append	ix 7.C The Lamperti Transformation 219
8	Estimation of Parameters of Stochastic Differential Equations 221
8.1	Introduction 221
8.2	The Transition Probability Density Function Is Known 222
8.3	The Transition Probability Density Function Is Unknown 227
8.3.1	Parameter Estimation via Approximation Methods 228
8.3.1.1	The EM Routine 228
8.3.1.2	The Ozaki Routine 230
8.3.1.3	The SO Routine 233
	ix 8.A The ML Technique 235
	ix 8.B The Log-Normal Probability Distribution 238
LL	0

Appendix 8.C The Markov Property, Transitional Densities, and the Likelihood Function of the Sample 239

Appendix 8.D Change of Variable 241

Appendix A: A Review of Some Fundamental Calculus Concepts

Appendix B: The Lebesgue Integral 259
Appendix C: Lebesgue–Stieltjes Integral 261
Appendix D: A Brief Review of Ordinary Differential Equations 263
References 275
Index 279