

Contents

1	Euclidean Geometry	1
1.1	Preliminaries	2
1.2	Distance Geometry	2
1.2.1	A Basic Formula	2
1.2.2	The Length of a Path	3
1.2.3	The First Variation Formula and Application to Billiards	4
1.3	Plane Curves	9
1.3.1	Length	9
1.3.2	Curvature	12
1.4	Global Theory of Closed Plane Curves	18
1.4.1	"Obvious" Truths About Curves Which are Hard to Prove	18
1.4.2	The Four Vertex Theorem	20
1.4.3	Convexity with Respect to Arc Length	22
1.4.4	Umlaufsatz with Corners	23
1.4.5	Heat Shrinking of Plane Curves	24
1.4.6	Arnol'd's Revolution in Plane Curve Theory	24
1.5	The Isoperimetric Inequality for Curves	26
1.6	The Geometry of Surfaces Before and After Gauß	29
1.6.1	Inner Geometry: a First Attempt	30
1.6.2	Looking for Shortest Curves: Geodesics	33
1.6.3	The Second Fundamental Form and Principal Curvatures	45
1.6.4	The Meaning of the Sign of K	52
1.6.5	Global Surface Geometry	55
1.6.6	Minimal Surfaces	58
1.6.7	The Hartman-Nirenberg Theorem for Inner Flat Surfaces	62
1.6.8	The Isoperimetric Inequality in \mathbb{E}^3 à la Gromov	63
1.6.8.1	Notes	66
1.7	Generic Surfaces	66
1.8	Heat and Wave Analysis in \mathbb{E}^2	70
1.8.1	Planar Physics	70
	1.8.1.1 Bibliographical Note	71

1.8.2	Why the Eigenvalue Problem?	71
1.8.3	Minimax	75
1.8.4	Shape of a Drum	78
1.8.4.1	A Few Direct Problems	79
1.8.4.2	The Faber-Krahn Inequality	81
1.8.4.3	Inverse Problems	83
1.8.5	Heat	87
1.8.5.1	Eigenfunctions	90
1.8.6	Relations Between the Two Spectra	91
1.9	Heat and Waves in $\mathbb{E}^3, \mathbb{E}^d$ and on the Sphere	94
1.9.1	Euclidean Spaces	94
1.9.2	Spheres	95
1.9.3	Billiards in Higher Dimensions	97
1.9.4	The Wave Equation Versus the Heat Equation	98
2	Transition	101
3	Surfaces from Gauß to Today	105
3.1	Gauß	105
3.1.1	Theorema Egregium	105
3.1.1.1	The First Proof of Gauß's Theorema Egregium; the Concept of ds^2	106
3.1.1.2	Second Proof of the Theorema Egregium	109
3.1.2	The Gauß-Bonnet Formula and the Rodrigues-Gauß Map	111
3.1.3	Parallel Transport	113
3.1.4	Inner Geometry	116
3.2	Alexandrov's Theorems	120
3.2.1	Angle Corrections of Legendre and Gauß in Geodesy	123
3.3	Cut Loci	125
3.4	Global Surface Theory	131
3.4.1	Bending Surfaces	131
3.4.1.1	Bending Polyhedra	132
3.4.1.2	Bending and Wrinkling with Little Smoothness	133
3.4.2	Mean Curvature Rigidity of the Sphere	134
3.4.3	Negatively Curved Surfaces	135
3.4.4	The Willmore Conjecture	136
3.4.5	The Global Gauß-Bonnet Theorem for Surfaces	136
3.4.6	The Hopf Index Formula	139

4 Riemann's Blueprints	143
4.1 Smooth Manifolds	143
4.1.1 Introduction	143
4.1.2 The Need for Abstract Manifolds	146
4.1.3 Examples	149
4.1.3.1 Submanifolds	151
4.1.3.2 Products	151
4.1.3.3 Lie Groups	152
4.1.3.4 Homogeneous Spaces	152
4.1.3.5 Grassmannians over Various Algebras	153
4.1.3.6 Gluing	156
4.1.4 The Classification of Manifolds	157
4.1.4.1 Surfaces	158
4.1.4.2 Higher Dimensions	159
4.1.4.3 Embedding Manifolds in Euclidean Space . .	161
4.2 Calculus on Manifolds	162
4.2.1 Tangent Spaces and the Tangent Bundle	162
4.2.2 Differential Forms and Exterior Calculus	166
4.3 Examples of Riemann's Definition	172
4.3.1 Riemann's Definition	172
4.3.2 Hyperbolic Geometry	176
4.3.3 Products, Coverings and Quotients	183
4.3.3.1 Products	183
4.3.3.2 Coverings	184
4.3.4 Homogeneous Spaces	186
4.3.5 Symmetric Spaces	189
4.3.5.1 Classification	192
4.3.5.2 Rank	193
4.3.6 Riemannian Submersions	194
4.3.7 Gluing and Surgery	196
4.3.7.1 Gluing of Hyperbolic Surfaces	196
4.3.7.2 Higher Dimensional Gluing	198
4.3.8 Classical Mechanics	199
4.4 The Riemann Curvature Tensor	200
4.4.1 Discovery and Definition	200
4.4.2 The Sectional Curvature	204
4.4.3 Standard Examples	207
4.4.3.1 Constant Sectional Curvature	207
4.4.3.2 Projective Spaces \mathbb{KP}^n	209
4.4.3.3 Products	209
4.4.3.4 Homogeneous Spaces	210
4.4.3.5 Hypersurfaces in Euclidean Space	211

4.5	A Naive Question: Does the Curvature Determine the Metric?	213
4.5.1	Surfaces	214
4.5.2	Any Dimension	215
4.6	Abstract Riemannian Manifolds	216
4.6.1	Isometrically Embedding Surfaces in \mathbb{E}^3	217
4.6.2	Local Isometric Embedding of Surfaces in \mathbb{E}^3	217
4.6.3	Isometric Embedding in Higher Dimensions	218
5	A One Page Panorama	219
6	Metric Geometry and Curvature	221
6.1	First Metric Properties	222
6.1.1	Local Properties	222
6.1.2	Hopf–Rinow and de Rham Theorems	226
6.1.2.1	Products	229
6.1.3	Convexity and Small Balls	229
6.1.4	Totally Geodesic Submanifolds	231
6.1.5	Center of Mass	233
6.1.6	Examples of Geodesics	235
6.1.7	Transition	238
6.2	First Technical Tools	239
6.3	Second Technical Tools	248
6.3.1	Exponential Map	248
6.3.1.1	Rank	250
6.3.2	Space Forms	251
6.3.3	Nonpositive Curvature	254
6.4	Triangle Comparison Theorems	257
6.4.1	Bounded Sectional Curvature	257
6.4.2	Ricci Lower Bound	262
6.4.3	Philosophy Behind These Bounds	267
6.5	Injectivity, Convexity Radius and Cut Locus	268
6.5.1	Definition of Cut Points and Injectivity Radius	268
6.5.2	Klingenberg and Cheeger Theorems	272
6.5.3	Convexity Radius	278
6.5.4	Cut Locus	278
6.5.5	Blaschke Manifolds	285
6.6	Geometric Hierarchy	286
6.6.1	The Geometric Hierarchy	289
6.6.1.1	Space Forms	289
6.6.1.2	Rank 1 Symmetric Spaces	289
6.6.1.3	Measure Isotropy	289
6.6.1.4	Symmetric Spaces	290
6.6.1.5	Homogeneous Spaces	290

6.6.2	Constant Sectional Curvature	290
6.6.2.1	Negatively Curved Space Forms in Three and Higher Dimensions	292
6.6.2.2	Mostow Rigidity	293
6.6.2.3	Classification of Arithmetic and Nonarithmetic Negatively Curved Space Forms	294
6.6.2.4	Volumes of Negatively Curved Space Forms	295
6.6.3	Rank 1 Symmetric Spaces	295
6.6.4	Higher Rank Symmetric Spaces	296
6.6.4.1	Superrigidity	296
6.6.5	Homogeneous Spaces	296
7	Volumes and Inequalities on Volumes of Cycles	299
7.1	Curvature Inequalities	299
7.1.1	Bounds on Volume Elements and First Applications .	299
7.1.1.1	The Canonical Measure	299
7.1.1.2	Volumes of Standard Spaces	303
7.1.1.3	The Isoperimetric Inequality for Spheres .	304
7.1.1.4	Sectional Curvature Upper Bounds	305
7.1.1.5	Ricci Curvature Lower Bounds	308
7.1.2	Isoperimetric Profile	315
7.1.2.1	Definition and Examples	315
7.1.2.2	The Gromov–Bérard–Besson–Gallot Bound .	319
7.1.2.3	Nonpositive Curvature on Noncompact Manifolds	322
7.2	Curvature Free Inequalities on Volumes of Cycles	325
7.2.1	Curves in Surfaces	325
7.2.1.1	Loewner, Pu and Blatter–Bavard Theorems	325
7.2.1.2	Higher Genus Surfaces	329
7.2.1.3	The Sphere	336
7.2.1.4	Homological Systoles	338
7.2.2	Inequalities for Curves	340
7.2.2.1	The Problem, and Standard Manifolds	340
7.2.2.2	Filling Volume and Filling Radius	342
7.2.2.3	Gromov’s Theorem and Sketch of the Proof	344
7.2.3	Higher Dimensional Systoles: Systolic Freedom Almost Everywhere	348
7.2.4	Embolic Inequalities	353
7.2.4.1	Introduction	353
7.2.4.2	The Unit Tangent Bundle	357
7.2.4.3	The Core of the Proof	359
7.2.4.4	Croke’s Three Results	363
7.2.4.5	Infinite Injectivity Radius	366
7.2.4.6	Using Embolic Inequalities	367

8 Transition: The Next Two Chapters	369
8.1 Spectral Geometry and Geodesic Dynamics	369
8.2 Why are Riemannian Manifolds So Important?	372
8.3 Positive Versus Negative Curvature	372
9 Spectrum of the Laplacian	373
9.1 History	374
9.2 Motivation	375
9.3 Setting Up	376
9.3.1 Xdefinition	376
9.3.2 The Hodge Star	378
9.3.3 Facts	380
9.3.4 Heat, Wave and Schrödinger Equations	381
9.4 Minimax	383
9.4.1 The Principle	383
9.4.2 An Application	385
9.5 Some Extreme Examples	387
9.5.1 Square Tori, Alias Several Variable Fourier Series	387
9.5.2 Other Flat Tori	388
9.5.3 Spheres	390
9.5.4 $\mathbb{K}\mathbb{P}^n$	390
9.5.5 Other Space Forms	391
9.6 Current Questions	392
9.6.1 Direct Questions About the Spectrum	392
9.6.2 Direct Problems About the Eigenfunctions	393
9.6.3 Inverse Problems on the Spectrum	393
9.7 First Tools: The Heat Kernel and Heat Equation	393
9.7.1 The Main Result	393
9.7.2 Great Hopes	396
9.7.3 The Heat Kernel and Ricci Curvature	401
9.8 The Wave Equation: The Gaps	402
9.9 The Wave Equation: Spectrum & Geodesic Flow	405
9.10 The First Eigenvalue	408
9.10.1 λ_1 and Ricci Curvature	408
9.10.2 Cheeger's Constant	409
9.10.3 λ_1 and Volume; Surfaces and Multiplicity	410
9.10.4 Kähler Manifolds	411
9.11 Results on Eigenfunctions	412
9.11.1 Distribution of the Eigenfunctions	412
9.11.2 Volume of the Nodal Hypersurfaces	413
9.11.3 Distribution of the Nodal Hypersurfaces	414
9.12 Inverse Problems	414
9.12.1 The Nature of the Image	414
9.12.2 Inverse Problems: Nonuniqueness	416
9.12.3 Inverse Problems: Finiteness, Compactness	418

9.12.4	Uniqueness and Rigidity Results	419
9.12.4.1	Vignéras Surfaces	420
9.13	Special Cases	421
9.13.1	Riemann Surfaces	421
9.13.2	Space Forms	424
9.13.2.1	Scars	426
9.14	The Spectrum of Exterior Differential Forms	426
10	Geodesic Dynamics	431
10.1	Introduction	432
10.2	Some Well Understood Examples	436
10.2.1	Surfaces of Revolution	436
10.2.1.1	Zoll Surfaces	436
10.2.1.2	Weinstein Surfaces	440
10.2.2	Ellipsoids and Morse Theory	440
10.2.3	Flat and Other Tori: Influence of the Fundamental Group	442
10.2.3.1	Flat Tori	442
10.2.3.2	Manifolds Which are not Simply Connected	443
10.2.3.3	Tori, not Flat	445
10.2.4	Space Forms	446
10.2.4.1	Space Form Surfaces	446
10.2.4.2	Higher Dimensional Space Forms	448
10.3	Geodesics Joining Two Points	449
10.3.1	Birkhoff's Proof for the Sphere	449
10.3.2	Morse Theory	453
10.3.3	Discoveries of Morse and Serre	454
10.3.4	Computing with Entropy	456
10.3.5	Rational Homology and Gromov's Work	458
10.4	Periodic Geodesics	461
10.4.1	The Difficulties	461
10.4.2	General Results	463
10.4.2.1	Gromoll and Meyer	463
10.4.2.2	Results for the Generic ("Bumpy") Case	465
10.4.3	Surfaces	466
10.4.3.1	The Lusternik–Schnirelmann Theorem	466
10.4.3.2	The Bangert–Franks–Hingston Results	468
10.5	The Geodesic Flow	471
10.5.1	Review of Ergodic Theory of Dynamical Systems	471
10.5.1.1	Ergodicity and Mixing	471
10.5.1.2	Notions of Entropy	473
10.6	Negative Curvature	478
10.6.1	Distribution of Geodesics	481
10.6.2	Distribution of Periodic Geodesics	481
10.7	Nonpositive Curvature	482

XVIII Contents

10.8	Entropies on Various Space Forms	483
10.8.1	Liouville Entropy	485
10.9	From Osserman to Lohkamp	485
10.10	Manifolds All of Whose Geodesics are Closed	488
10.10.1	Definitions and Caution	488
10.10.2	Bott and Samelson Theorems	490
10.10.3	The Structure on a Given S^d and \mathbb{KP}^n	492
10.11	Inverse Problems: Conjugacy of Geodesic Flows	495
11	Best Metrics	499
11.1	Introduction and a Possible Approach	499
11.1.1	An Approach	501
11.2	Purely Geometric Functionals	503
11.2.1	Systolic Inequalities	503
11.2.2	Counting Periodic Geodesics	504
11.2.3	The Embolic Constant	504
11.2.4	Diameter and Injectivity	505
11.3	Least curved	506
11.3.1	Definitions	506
11.3.1.1	$\inf \ R\ _{L^{d/2}}$	506
11.3.1.2	Minimal Volume	507
11.3.1.3	Minimal Diameter	507
11.3.2	The Case of Surfaces	508
11.3.3	Generalities, Compactness, Finiteness and Equivalence	509
11.3.4	Manifolds with $\inf \text{Vol}$ (resp. $\inf \ R\ _{L^{d/2}}, \inf \text{diam}) = 0$	511
11.3.4.1	Circle Fibrations and Other Examples	511
11.3.4.2	Alloff–Wallach’s Type of Examples	513
11.3.4.3	Nilmanifolds and the Converse: Almost Flat Manifolds	514
11.3.4.4	The Examples of Cheeger and Rong	514
11.3.5	Some Manifolds with $\inf \text{Vol} > 0$ and $\inf \ R\ _{L^{d/2}} > 0$	515
11.3.5.1	Using Integral Formulas	515
11.3.5.2	The Simplicial Volume of Gromov	516
11.3.6	$\inf \ R\ _{L^{d/2}}$ in Four Dimensions	518
11.3.7	Summing up Questions on $\inf \text{Vol}, \inf \ R\ _{L^{d/2}}$	519
11.4	Einstein Manifolds	520
11.4.1	Hilbert’s Variational Principle and Great Hopes	520
11.4.2	The Examples from the Geometric Hierarchy	524
11.4.2.1	Symmetric Spaces	524
11.4.2.2	Homogeneous Spaces and Others	524
11.4.3	Examples from Analysis: Evolution by Ricci Flow	525

11.4.4	Examples from Analysis: Kähler Manifolds	526
11.4.5	The Sporadic Examples	528
11.4.6	Around Existence and Uniqueness	529
11.4.6.1	Existence	529
11.4.6.2	Uniqueness	530
11.4.6.3	Moduli	531
11.4.6.4	The Set of Constants, Ricci Flat Metrics	532
11.4.7	The Yamabe Problem	533
11.5	The Bewildering Fractal Landscape of $\mathcal{RS}(M)$ According to Nabutovsky	534
12	From Curvature to Topology	543
12.1	Some History, and Structure of the Chapter	543
12.1.1	Hopf's Inspiration	543
12.1.2	Hierarchy of Curvatures	546
12.1.2.1	Control via Curvature	546
12.1.2.2	Other Curvatures	547
12.1.2.3	The Problem of Rough Classification	548
12.1.2.4	References on the Topic, and the Significance of Noncompact Manifolds	549
12.2	Pinching Problems	549
12.2.1	Introduction	549
12.2.2	Positive Pinching	552
12.2.2.1	The Sphere Theorem	552
12.2.2.2	Sphere Theorems Invoking Bounds on Other Invariants	557
12.2.2.3	Homeomorphic Pinching	558
12.2.2.4	The Sphere Theorem with Lower Bound on Diameter, and no Upper Bound on Curvature	562
12.2.2.5	Topology at the Diameter Pinching Limit	565
12.2.2.6	Pointwise Pinching	567
12.2.2.7	Cutting Down the Hypotheses	567
12.2.3	Pinching Near Zero	568
12.2.4	Negative Pinching	569
12.2.5	Ricci Curvature Pinching	571
12.3	Curvature of Fixed Sign	576
12.3.1	The Positive Side: Sectional Curvature	576
12.3.1.1	The Known Examples	576
12.3.1.2	Homology Type and the Fundamental Group	580
12.3.1.3	The Noncompact Case	583
12.3.1.4	Positivity of the Curvature Operator	588
12.3.1.5	Possible Approaches, Looking to the Future	590

12.3.2	Ricci Curvature: Positive, Negative and Just Below	593
12.3.3	The Positive Side: Scalar Curvature	599
12.3.3.1	The Hypersurfaces of Schoen & Yau	600
12.3.3.2	Geometrical Descriptions	601
12.3.3.3	Gromov's Quantization of K -theory and Topological Implications of Positive Scalar Curvature	602
12.3.3.4	Trichotomy	603
12.3.3.5	The Proof	603
12.3.3.6	The Gromov–Lawson Torus Theorem	604
12.3.4	The Negative Side: Sectional Curvature	605
12.3.4.1	Introduction	605
12.3.4.2	Literature	605
12.3.4.3	Quasi-isometries	606
12.3.4.4	Volume and Fundamental Group	609
12.3.4.5	Negative Versus Nonpositive Curvature	612
12.3.5	The Negative Side: Ricci Curvature	613
12.4	Finiteness and Collapsing	614
12.4.1	Finiteness	614
12.4.1.1	Cheeger's Finiteness Theorems	614
12.4.1.2	More Finiteness Theorems	618
12.4.1.3	Ricci Curvature	622
12.4.2	Compactness and Convergence	624
12.4.2.1	Motivation	624
12.4.2.2	History	624
12.4.2.3	Contemporary Definitions and Results	625
12.4.3	Collapsing and the Space of Riemannian Metrics	630
12.4.3.1	Collapsing	630
12.4.3.2	Closures on a Compact Manifold	634
13	Holonomy Groups and Kähler Manifolds	637
13.1	Definitions and Philosophy	637
13.2	Examples	639
13.3	General Structure Theorems	641
13.4	Classification	643
13.5	The Rare Cases	646
13.5.1	G_2 and $\text{Spin}(7)$	646
13.5.2	Quaternionic Kähler Manifolds	647
13.5.2.1	The Bérard Bergery/Salamon Twistor Space of Quaternionic Kähler Manifolds	649
13.5.2.2	The Konishi Twistor Space of a Quaternionic Kähler Manifold	651
13.5.2.3	Other Twistor Spaces	651
13.5.3	Ricci Flat Kähler and Hyper-Kähler Manifolds	652
13.5.3.1	Hyperkähler Manifolds	652

13.6	Kähler Manifolds	654
13.6.1	Symplectic Structures on Kähler Manifolds	655
13.6.2	Imitating Complex Algebraic Geometry on Kähler Manifolds	655
14	Some Other Important Topics	659
14.1	Noncompact Manifolds	660
14.1.1	Noncompact Manifolds of Nonnegative Ricci Curvature	660
14.1.2	Finite Volume	661
14.1.3	Bounded Geometry	661
14.1.4	Harmonic Functions	662
14.1.5	Structure at Infinity	662
14.1.6	Chopping	662
14.1.7	Positive Mass	662
14.1.8	Cohomology and Homology Theories	663
14.2	Bundles over Riemannian Manifolds	663
14.2.1	Differential Forms and Related Bundles	663
14.2.1.1	The Hodge Star	664
14.2.1.2	A Variational Problem for Differential Forms and the Laplace Operator	664
14.2.1.3	Calibration	666
14.2.1.4	Harmonic Analysis of Other Tensors	667
14.2.2	Spinors	668
14.2.2.1	Algebra of Spinors	668
14.2.2.2	Spinors on Riemannian Manifolds	669
14.2.2.3	History of Spinors	669
14.2.2.4	Applying Spinors	670
14.2.2.5	Warning: Beware of Harmonic Spinors	670
14.2.2.6	The Half Pontryagin Class	670
14.2.2.7	Reconstructing the Metric from the Dirac Operator	671
14.2.2.8	Spin ^c Structures	671
14.2.3	Various Other Bundles	671
14.2.3.1	Secondary Characteristic Classes	672
14.2.3.2	Yang–Mills Theory	672
14.2.3.3	Twistor Theory	672
14.2.3.4	K-theory	673
14.2.3.5	The Atiyah–Singer Index Theorem	674
14.2.3.6	Supersymmetry and Supergeometry	674
14.3	Harmonic Maps Between Riemannian Manifolds	674
14.4	Low Dimensional Riemannian Geometry	676
14.5	Some Generalizations of Riemannian Geometry	676
14.5.1	Boundaries	676
14.5.2	Orbifolds	677

XXII Contents

14.5.3	Conical Singularities	678
14.5.4	Spectra of Singular Spaces	678
14.5.5	Alexandrov Spaces	678
14.5.6	CAT Spaces	680
14.5.6.1	The CAT (k) Condition	680
14.5.7	Carnot–Carathéodory Spaces	681
14.5.7.1	Example: the Heisenberg Group	681
14.5.8	Finsler Geometry	682
14.5.9	Riemannian Foliations	683
14.5.10	Pseudo-Riemannian Manifolds	683
14.5.11	Infinite Dimensional Riemannian Geometry	684
14.5.12	Noncommutative Geometry	685
14.6	Gromov's mm Spaces	685
14.7	Submanifolds	690
14.7.1	Higher Dimensions	690
14.7.2	Geometric Measure Theory and Pseudoholomorphic Curves	691
15	The Technical Chapter	693
15.1	Vector Fields and Tensors	693
15.2	Tensors Dual via the Metric: Index Aerobics	696
15.3	The Connection, Covariant Derivative and Curvature	697
15.4	Parallel Transport	701
15.4.1	Curvature from Parallel Transport	703
15.5	Absolute (Ricci) Calculus and Commutation Formulas: Index Gymnastics	704
15.6	Hodge and the Laplacian, Bochner's Technique	706
15.6.1	Bochner's Technique for Higher Degree Differential Forms	708
15.7	Gauß–Bonnet–Chern	709
15.7.1	Chern's Proof of Gauß–Bonnet for Surfaces	710
15.7.2	The Proof of Allendoerfer and Weil	711
15.7.3	Chern's Proof in all Even Dimensions	713
15.7.4	Chern Classes of Vector Bundles	714
15.7.5	Pontryagin Classes	715
15.7.6	The Euler Class	715
15.7.7	The Absence of Other Characteristic Classes	716
15.7.8	Applying Characteristic Classes	716
15.7.9	Characteristic Numbers	716
15.8	Examples of Curvature Calculations	718
15.8.1	Homogeneous Spaces	718
15.8.2	Riemannian Submersions	719

References	723
Acknowledgements	789
List of Notation	791
List of Authors	797
Subject Index	811

Contents

1.1 Preliminaries	3
1.2 Distance Geometry	3
1.2.1 A Mean Formula	2
1.2.2 The Length of a Path	3
1.2.3 The First Variation Formula and Application to Biharis	4
1.3 Plane Curves	9
1.3.1 Length	9
1.3.2 Curvature	12
1.4 Global Theory of Closed Plane Curves	18
1.4.1 "Known" Truths About Curves Which are Hard to Prove	18
1.4.2 The Four Vertex Theorem	20
1.4.3 Convexity with Respect to Arc Length	22
1.4.4 Meissner with Corners	23
1.4.5 Area Shrinking of Plane Curves	24
1.4.6 Arnold's Revolution in Plane Curve Theory	24
1.5 The Isoperimetric Inequality for Curves	26
1.6 The Geometry of Surfaces Before and After Gauss	29
1.6.1 Inner Geometry: a First Attempt	30
1.6.2 Curves for Shortest Curves: Geodesics	33
1.6.3 The Second Fundamental Form and Principal Curvatures	45
1.6.4 The Meaning of the Sign of K	52
1.6.5 Global Surface Geometry	55
1.6.6 Minimal Surfaces	56
1.6.7 The Hartman-Nirenberg Theorem for Super-Flat Surfaces	62
1.6.8 The Isoperimetric Inequality in \mathbb{H}^2 A la Gromov	63
1.7 Geometric Surfaces	66
1.8 Heat and Wave Analysis in E^d	70
1.8.1 Planar Physics	70
1.8.2 Why the Eigenvalue Problem	71
1.8.3 Minimax	73