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Introduction

Topology is one of the better-known areas of modern mathematics. Most people have heard the statement that a topologist is someone who cannot tell the difference between a tea cup and a doughnut. This is true, and we will see why in Chapter 2. Clearly, then, topology ignores some things and perceives similarities between apparently dissimilar objects. As we will discover, the key to what topology ignores, and to what it concentrates on, is the behaviour of continuous functions. Topology studies the ways in which the properties of the domain and range constrain the behaviour of a continuous function.

For example, every continuous integer-valued function on the real line, \mathbb{R} , is constant. We do not need to know anything about the function apart from the fact that its domain and its range is \mathbb{Z} , the set of integers. Then, somehow, the topology of \mathbb{R} and \mathbb{Z} force that function to be constant. What matters here is the topology of \mathbb{R} and \mathbb{Z} .

Another example is the fact that any continuous real function defined on the closed interval $[0, 1]$ (of all real numbers x such that $0 \leq x \leq 1$) must be bounded, that is, there are real numbers j, k such that $j < f(x) < k$ for all $x \in [0, 1]$. However, a continuous function defined on the open interval $(0, 1)$ (of all real numbers x such that $0 < x < 1$) need not be bounded. The function $f(x) = 1/x$ is an example of an unbounded function on $(0, 1)$. So something about the topology of $[0, 1]$ forces functions to be bounded whereas $(0, 1)$ does not. Again, it is the topology of \mathbb{R} , and the topology of $(0, 1)$ which produce this different behaviour.

Even in complex analysis there is a very clear example of topology in action. The unit disc is a possible region, and a complex-valued function defined