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by P. Jónsson and H. Rose

In this chapter we discuss some of the more recent results and give a general overview of what is currently known about lattice varieties. Of course it is impossible to give a comprehensive account. Often we only cite recent or survey papers, which themselves have many more references. We would like to apologize in advance for any errors, omissions, or misquoting of results.

For proofs of the results mentioned here, we refer the reader to the original papers. Details of many of the results from before 1992 can also be found in our monograph, P. Jónsson and H. Rose [250].

1-1. The lattice \mathbf{A}

Recall from [L1], Section VI.2, that the lattice \mathbf{A} of all lattice varieties is a dually algebraic, distributive lattice that has the variety \mathbf{L} of all lattices at the top, the variety \mathbf{T} of all trivial lattices at the bottom, and the variety $\mathbf{D} = \text{Var}(\mathbf{C}_2)$ of all distributive lattices as the unique atom. To exclude that \mathbf{L} is join-irreducible and has no coatoms, B. Jónsson [255] argued as follows: Let \mathbf{V}, \mathbf{W} be proper subvarieties of \mathbf{L} and choose lattices $\mathbf{K} \notin \mathbf{V}$, $\mathbf{L} \notin \mathbf{W}$. Using P.M. Whitman's [450] result that every lattice can be embedded in a partition lattice, one obtains a subdirectly irreducible lattice \mathbf{S} that extends $\mathbf{K} \times \mathbf{L}$. Since $\mathbf{S} \notin \text{St}(\mathbf{V}) \vee \text{St}(\mathbf{W}) = \text{St}(\mathbf{V} \vee \mathbf{W})$ (where $\text{St}(\mathbf{V})$ denotes the