

# Contents

*There is no philosophy which is not founded upon knowledge of  
the phenomena, but to get any profit from this knowledge  
it is absolutely necessary  
to be a mathematician.* Daniel Bernoulli, 1700–1782

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