



# LATTICES and ORDERED SETS

Steven Roman

This book is intended to be a thorough introduction to the subject of ordered sets and lattices, with an emphasis on the latter. It can be used for a course at the graduate or advanced undergraduate level or for independent study. Prerequisites are kept to a minimum, but an introductory course in abstract algebra is highly recommended, since many of the examples are drawn from this area.

The book has an excellent choice of topics, including a chapter on well ordering and ordinal numbers, which is not usually found in other texts. The approach is user-friendly and the presentation is lucid. There are more than 240 carefully chosen exercises.

Topic coverage includes: modular, semimodular and distributive lattices, boolean algebras, representation of distributive lattices, algebraic lattices, congruence relations on lattices, free lattices, fixed-point theorems, duality theory and more.

Steven Roman is the author of many successful textbooks, including *Advanced Linear Algebra*, 3rd Edition (Springer 2007), *Field Theory*, 2nd Edition (Springer 2005), and *Introduction to the Mathematics of Finance* (2004).

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