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of Finite Groups by J.-P. Serre ([Ser77]); and among this there are a few books are concentrated on the case of symmetric groups, for example *Symmetric Group Representations, Combinatorial Algorithms and Symmetric Functions* by R. Sagali (see [Sag01]). The point of view and interest of the present book is the following: we shall show that most of the calculations on symmetric groups can be performed, or at least eased by using some appropriate algebras of functions. It is well known since the works of Frobenius and Schur that the algebra of symmetric functions encodes most of the theory of characters of symmetric groups. In this book, we shall use the algebra of symmetric functions as the starting point of the representation theory of symmetric groups, and then go forward by introducing other interesting algebras, such as:

- the algebra of observables of partitions, originally called “polynomial functions on Young diagrams,” and whose construction is due to Kerov and Olshanski;
- the Hopf algebras of non-commutative symmetric functions, quasi-symmetric functions and free quasi-symmetric functions, which contain and generalize the algebra of symmetric functions.

This algebraic approach to the representation theory of symmetric groups can be opposed to a more traditional approach which is of combinatorial nature, and which gives a large role to the famous Young tableaux. The approach with algebras of functions has several advantages:

1. First, if one tries to replace the symmetric group by finite-dimensional algebras related to it (the so-called partition algebras, or the Hecke algebras), then one can still use the algebra of symmetric functions to treat the character theory of these algebras, and in this setting, most of the results related to the symmetric groups have direct analogues. In this book, we shall treat the case of Hecke algebras, which is a good example of this kind of extension of the theory of symmetric groups (the case of partition algebras is treated for instance in a recent book by Ceccherini-Silberstein, Scarabotti and Tollu, see