

# Contents

<i>Preface</i>		xv
<i>Acknowledgements</i>		xxiii
<i>Notation</i>		xxvi
<i>Prologue</i>		1
1 The roots of science		7
1.1 The quest for the forces that shape the world		7
1.2 Mathematical truth		9
1.3 Is Plato's mathematical world 'real'?		12
1.4 Three worlds and three deep mysteries		17
1.5 The Good, the True, and the Beautiful		22
2 An ancient theorem and a modern question		25
2.1 The Pythagorean theorem		25
2.2 Euclid's postulates		28
2.3 Similar-areas proof of the Pythagorean theorem		31
2.4 Hyperbolic geometry: conformal picture		33
2.5 Other representations of hyperbolic geometry		37
2.6 Historical aspects of hyperbolic geometry		42
2.7 Relation to physical space		46
3 Kinds of number in the physical world		51
3.1 A Pythagorean catastrophe?		51
3.2 The real-number system		54
3.3 Real numbers in the physical world		59
3.4 Do natural numbers need the physical world?		63
3.5 Discrete numbers in the physical world		65
4 Magical complex numbers		71
4.1 The magic number 'i'		71
4.2 Solving equations with complex numbers		74

4.3	Convergence of power series	76
4.4	Caspar Wessel's complex plane	81
4.5	How to construct the Mandelbrot set	83
5	Geometry of logarithms, powers, and roots	86
5.1	Geometry of complex algebra	86
5.2	The idea of the complex logarithm	90
5.3	Multiple valuedness, natural logarithms	92
5.4	Complex powers	96
5.5	Some relations to modern particle physics	100
6	Real-number calculus	103
6.1	What makes an honest function?	103
6.2	Slopes of functions	105
6.3	Higher derivatives; $C^\infty$ -smooth functions	107
6.4	The 'Eulerian' notion of a function?	112
6.5	The rules of differentiation	114
6.6	Integration	116
7	Complex-number calculus	122
7.1	Complex smoothness; holomorphic functions	122
7.2	Contour integration	123
7.3	Power series from complex smoothness	127
7.4	Analytic continuation	129
8	Riemann surfaces and complex mappings	135
8.1	The idea of a Riemann surface	135
8.2	Conformal mappings	138
8.3	The Riemann sphere	142
8.4	The genus of a compact Riemann surface	145
8.5	The Riemann mapping theorem	148
9	Fourier decomposition and hyperfunctions	153
9.1	Fourier series	153
9.2	Functions on a circle	157
9.3	Frequency splitting on the Riemann sphere	161
9.4	The Fourier transform	164
9.5	Frequency splitting from the Fourier transform	166
9.6	What kind of function is appropriate?	168
9.7	Hyperfunctions	172

10	Surfaces	179
10.1	Complex dimensions and real dimensions	179
10.2	Smoothness, partial derivatives	181
10.3	Vector fields and 1-forms	185
10.4	Components, scalar products	190
10.5	The Cauchy–Riemann equations	193
11	Hypercomplex numbers	198
11.1	The algebra of quaternions	198
11.2	The physical role of quaternions?	200
11.3	Geometry of quaternions	203
11.4	How to compose rotations	206
11.5	Clifford algebras	208
11.6	Grassmann algebras	211
12	Manifolds of $n$ dimensions	217
12.1	Why study higher-dimensional manifolds?	217
12.2	Manifolds and coordinate patches	221
12.3	Scalars, vectors, and covectors	223
12.4	Grassmann products	227
12.5	Integrals of forms	229
12.6	Exterior derivative	231
12.7	Volume element; summation convention	237
12.8	Tensors; abstract-index and diagrammatic notation	239
12.9	Complex manifolds	243
13	Symmetry groups	247
13.1	Groups of transformations	247
13.2	Subgroups and simple groups	250
13.3	Linear transformations and matrices	254
13.4	Determinants and traces	260
13.5	Eigenvalues and eigenvectors	263
13.6	Representation theory and Lie algebras	266
13.7	Tensor representation spaces; reducibility	270
13.8	Orthogonal groups	275
13.9	Unitary groups	281
13.10	Symplectic groups	286
14	Calculus on manifolds	292
14.1	Differentiation on a manifold?	292
14.2	Parallel transport	294
14.3	Covariant derivative	298
14.4	Curvature and torsion	301

14.5	Geodesics, parallelograms, and curvature	303
14.6	Lie derivative	309
14.7	What a metric can do for you	317
14.8	Symplectic manifolds	321
15	Fibre bundles and gauge connections	325
15.1	Some physical motivations for fibre bundles	325
15.2	The mathematical idea of a bundle	328
15.3	Cross-sections of bundles	331
15.4	The Clifford-Hopf bundle	334
15.5	Complex vector bundles, (co)tangent bundles	338
15.6	Projective spaces	341
15.7	Non-triviality in a bundle connection	345
15.8	Bundle curvature	349
16	The ladder of infinity	357
16.1	Finite fields	357
16.2	A finite or infinite geometry for physics?	359
16.3	Different sizes of infinity	364
16.4	Cantor's diagonal slash	367
16.5	Puzzles in the foundations of mathematics	371
16.6	Turing machines and Gödel's theorem	374
16.7	Sizes of infinity in physics	378
17	Spacetime	383
17.1	The spacetime of Aristotelian physics	383
17.2	Spacetime for Galilean relativity	385
17.3	Newtonian dynamics in spacetime terms	388
17.4	The principle of equivalence	390
17.5	Cartan's 'Newtonian spacetime'	394
17.6	The fixed finite speed of light	399
17.7	Light cones	401
17.8	The abandonment of absolute time	404
17.9	The spacetime for Einstein's general relativity	408
18	Minkowskian geometry	412
18.1	Euclidean and Minkowskian 4-space	412
18.2	The symmetry groups of Minkowski space	415
18.3	Lorentzian orthogonality; the 'clock paradox'	417
18.4	Hyperbolic geometry in Minkowski space	422
18.5	The celestial sphere as a Riemann sphere	428
18.6	Newtonian energy and (angular) momentum	431
18.7	Relativistic energy and (angular) momentum	434

19	The classical fields of Maxwell and Einstein	440
19.1	Evolution away from Newtonian dynamics	440
19.2	Maxwell's electromagnetic theory	442
19.3	Conservation and flux laws in Maxwell theory	446
19.4	The Maxwell field as gauge curvature	449
19.5	The energy-momentum tensor	455
19.6	Einstein's field equation	458
19.7	Further issues: cosmological constant; Weyl tensor	462
19.8	Gravitational field energy	464
20	Lagrangians and Hamiltonians	471
20.1	The magical Lagrangian formalism	471
20.2	The more symmetrical Hamiltonian picture	475
20.3	Small oscillations	478
20.4	Hamiltonian dynamics as symplectic geometry	483
20.5	Lagrangian treatment of fields	486
20.6	How Lagrangians drive modern theory	489
21	The quantum particle	493
21.1	Non-commuting variables	493
21.2	Quantum Hamiltonians	496
21.3	Schrödinger's equation	498
21.4	Quantum theory's experimental background	500
21.5	Understanding wave-particle duality	505
21.6	What is quantum 'reality'?	507
21.7	The 'holistic' nature of a wavefunction	511
21.8	The mysterious 'quantum jumps'	516
21.9	Probability distribution in a wavefunction	517
21.10	Position states	520
21.11	Momentum-space description	521
22	Quantum algebra, geometry, and spin	527
22.1	The quantum procedures $\mathbf{U}$ and $\mathbf{R}$	527
22.2	The linearity of $\mathbf{U}$ and its problems for $\mathbf{R}$	530
22.3	Unitary structure, Hilbert space, Dirac notation	533
22.4	Unitary evolution: Schrödinger and Heisenberg	535
22.5	Quantum 'observables'	538
22.6	YES/NO measurements; projectors	542
22.7	Null measurements; helicity	544
22.8	Spin and spinors	549
22.9	The Riemann sphere of two-state systems	553
22.10	Higher spin: Majorana picture	559
22.11	Spherical harmonics	562

22.12	Relativistic quantum angular momentum	566
22.13	The general isolated quantum object	570
23	The entangled quantum world	578
23.1	Quantum mechanics of many-particle systems	578
23.2	Hugeness of many-particle state space	580
23.3	Quantum entanglement; Bell inequalities	582
23.4	Bohm-type EPR experiments	585
23.5	Hardy's EPR example: almost probability-free	589
23.6	Two mysteries of quantum entanglement	591
23.7	Bosons and fermions	594
23.8	The quantum states of bosons and fermions	596
23.9	Quantum teleportation	598
23.10	Quanglement	603
24	Dirac's electron and antiparticles	609
24.1	Tension between quantum theory and relativity	609
24.2	Why do antiparticles imply quantum fields?	610
24.3	Energy positivity in quantum mechanics	612
24.4	Difficulties with the relativistic energy formula	614
24.5	The non-invariance of $\partial/\partial t$	616
24.6	Clifford–Dirac square root of wave operator	618
24.7	The Dirac equation	620
24.8	Dirac's route to the positron	622
25	The standard model of particle physics	627
25.1	The origins of modern particle physics	627
25.2	The zigzag picture of the electron	628
25.3	Electroweak interactions; reflection asymmetry	632
25.4	Charge conjugation, parity, and time reversal	638
25.5	The electroweak symmetry group	640
25.6	Strongly interacting particles	645
25.7	'Coloured quarks'	648
25.8	Beyond the standard model?	651
26	Quantum field theory	655
26.1	Fundamental status of QFT in modern theory	655
26.2	Creation and annihilation operators	657
26.3	Infinite-dimensional algebras	660
26.4	Antiparticles in QFT	662
26.5	Alternative vacua	664
26.6	Interactions: Lagrangians and path integrals	665
26.7	Divergent path integrals: Feynman's response	670
26.8	Constructing Feynman graphs; the S-matrix	672
26.9	Renormalization	675

26.10	Feynman graphs from Lagrangians	680
26.11	Feynman graphs and the choice of vacuum	681
27	The Big Bang and its thermodynamic legacy	686
27.1	Time symmetry in dynamical evolution	686
27.2	Submicroscopic ingredients	688
27.3	Entropy	690
27.4	The robustness of the entropy concept	692
27.5	Derivation of the second law—or not?	696
27.6	Is the whole universe an ‘isolated system’?	699
27.7	The role of the Big Bang	702
27.8	Black holes	707
27.9	Event horizons and spacetime singularities	712
27.10	Black-hole entropy	714
27.11	Cosmology	717
27.12	Conformal diagrams	723
27.13	Our extraordinarily special Big Bang	726
28	Speculative theories of the early universe	735
28.1	Early-universe spontaneous symmetry breaking	735
28.2	Cosmic topological defects	739
28.3	Problems for early-universe symmetry breaking	742
28.4	Inflationary cosmology	746
28.5	Are the motivations for inflation valid?	753
28.6	The anthropic principle	757
28.7	The Big Bang’s special nature: an anthropic key?	762
28.8	The Weyl curvature hypothesis	765
28.9	The Hartle–Hawking ‘no-boundary’ proposal	769
28.10	Cosmological parameters: observational status?	772
29	The measurement paradox	782
29.1	The conventional ontologies of quantum theory	782
29.2	Unconventional ontologies for quantum theory	785
29.3	The density matrix	791
29.4	Density matrices for spin $\frac{1}{2}$ : the Bloch sphere	793
29.5	The density matrix in EPR situations	797
29.6	FAPP philosophy of environmental decoherence	802
29.7	Schrödinger’s cat with ‘Copenhagen’ ontology	804
29.8	Can other conventional ontologies resolve the ‘cat’?	806
29.9	Which unconventional ontologies may help?	810
30	Gravity’s role in quantum state reduction	816
30.1	Is today’s quantum theory here to stay?	816
30.2	Clues from cosmological time asymmetry	817

30.3	Time-asymmetry in quantum state reduction	819
30.4	Hawking's black-hole temperature	823
30.5	Black-hole temperature from complex periodicity	827
30.6	Killing vectors, energy flow—and time travel!	833
30.7	Energy outflow from negative-energy orbits	836
30.8	Hawking explosions	838
30.9	A more radical perspective	842
30.10	Schrödinger's lump	846
30.11	Fundamental conflict with Einstein's principles	849
30.12	Preferred Schrödinger–Newton states?	853
30.13	FELIX and related proposals	856
30.14	Origin of fluctuations in the early universe	861
<b>31</b>	<b>Supersymmetry, supra-dimensionality, and strings</b>	<b>869</b>
31.1	Unexplained parameters	869
31.2	Supersymmetry	873
31.3	The algebra and geometry of supersymmetry	877
31.4	Higher-dimensional spacetime	880
31.5	The original hadronic string theory	884
31.6	Towards a string theory of the world	887
31.7	String motivation for extra spacetime dimensions	890
31.8	String theory as quantum gravity?	892
31.9	String dynamics	895
31.10	Why don't we see the extra space dimensions?	897
31.11	Should we accept the quantum-stability argument?	902
31.12	Classical instability of extra dimensions	905
31.13	Is string QFT finite?	907
31.14	The magical Calabi–Yau spaces; M-theory	910
31.15	Strings and black-hole entropy	916
31.16	The 'holographic principle'	920
31.17	The D-brane perspective	923
31.18	The physical status of string theory?	926
<b>32</b>	<b>Einstein's narrower path; loop variables</b>	<b>934</b>
32.1	Canonical quantum gravity	934
32.2	The chiral input to Ashtekar's variables	935
32.3	The form of Ashtekar's variables	938
32.4	Loop variables	941
32.5	The mathematics of knots and links	943
32.6	Spin networks	946
32.7	Status of loop quantum gravity?	952
<b>33</b>	<b>More radical perspectives; twistor theory</b>	<b>958</b>
33.1	Theories where geometry has discrete elements	958
33.2	Twistors as light rays	962

33.3	Conformal group; compactified Minkowski space	968
33.4	Twistors as higher-dimensional spinors	972
33.5	Basic twistor geometry and coordinates	974
33.6	Geometry of twistors as spinning massless particles	978
33.7	Twistor quantum theory	982
33.8	Twistor description of massless fields	985
33.9	Twistor sheaf cohomology	987
33.10	Twistors and positive/negative frequency splitting	993
33.11	The non-linear graviton	995
33.12	Twistors and general relativity	1000
33.13	Towards a twistor theory of particle physics	1001
33.14	The future of twistor theory?	1003
34	Where lies the road to reality?	1010
34.1	Great theories of 20th century physics—and beyond?	1010
34.2	Mathematically driven fundamental physics	1014
34.3	The role of fashion in physical theory	1017
34.4	Can a wrong theory be experimentally refuted?	1020
34.5	Whence may we expect our next physical revolution?	1024
34.6	What is reality?	1027
34.7	The roles of mentality in physical theory	1030
34.8	Our long mathematical road to reality	1033
34.9	Beauty and miracles	1038
34.10	Deep questions answered, deeper questions posed	1043
	<i>Epilogue</i>	1048
	<i>Bibliography</i>	1050
	<i>Index</i>	1086

The reader will find that in this book I have not shied away from presenting mathematical formulas, despite dire warnings of the severe reduction in readership that this will entail. I have thought seriously about this question, and have come to the conclusion that what I have to say cannot reasonably be conveyed without a certain amount of mathematical notation and the exploration of genuine mathematical concepts. The understanding that we have of the principles that actually underlie the behaviour of our physical world indeed depends upon some appreciation of its mathematics. Some people might take this as a cause for despair, as they will have formed the belief that they have no capacity for mathematics, no matter at how elementary a level. How could it be possible, they might well argue, for them to comprehend the research going on at the cutting edge of physical theory if they cannot even master the manipulation of fractions? Well, I certainly see the difficulty.