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The importance of Markov chains comes from two facts: (i) there are a large number of physical, biological, economic, and social phenomena that can be modeled in this way, and (ii) there is a well-developed theory that allows us to do computations. We begin with a famous example, then describe the property that is the defining feature of Markov chains.

Example 1.1 (Gambler's Ruin). Consider a gambling game in which on any turn you win \$1 with probability $p = 0.4$ or lose \$1 with probability $1 - p = 0.6$. Suppose further that you adopt the rule that you quit playing if your fortune reaches 0. Of course, if your fortune reaches \$N the casino makes you stop.

Let X_n be the amount of money you have after n plays. Your fortune, X_n , has the "Markov property." In words, this means that given the current state, X_n , any other information about the past is irrelevant for predicting the next state X_{n+1} . To check this for the gambler's ruin chain, we note that if you are still playing at time n , i.e., your fortune $X_n = i$ with $0 < i < N$, then for any possible history of your wealth $\{X_{n-1}, X_{n-2}, \dots, X_1, X_0\}$

$$P(X_{n+1} = i + 1 | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = 0.4$$

you are to increase your wealth by one unit you have to win your next bet. Here we have used $P(B|A)$ for the conditional probability of the event B given that A occurs. Recall that this is defined by

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

If you need help with this notion, see Sect. A.1 of the appendix.