Contents

1 Markov Chains 1
1.1 Definitions and Examples 1
1.2 Multistep Transition Probabilities 9
1.3 Classification of States 13
1.4 Stationary Distributions 21
1.4.1 Doubly Stochastic Chains 26
1.5 Detailed Balance Condition 28
1.5.1 Reversibility 34
1.5.2 The Metropolis–Hastings Algorithm 35
1.5.3 Kolmogorow Cycle Condition 38
1.6 Limit Behavior 40
1.7 Returns to a Fixed State 46
1.8 Proof of the Convergence Theorem* 50
1.9 Exit Distributions 53
1.10 Exit Times 61
1.11 Infinite State Spaces* 68
1.12 Chapter Summary 74
1.13 Exercises 78

2 Poisson Processes 95
2.1 Exponential Distribution 95
2.2 Defining the Poisson Process 100
2.2.1 Constructing the Poisson Process 103
2.2.2 More Realistic Models 104
2.3 Compound Poisson Processes 106
2.4 Transformations 108
2.4.1 Thinning 108
2.4.2 Superposition 112
2.4.3 Conditioning 113
2.5 Chapter Summary 115
2.6 Exercises 116

3 Renewal Processes ... 125
3.1 Laws of Large Numbers ... 125
3.2 Applications to Queueing Theory ... 130
3.2.1 GI/G/1 Queue ... 130
3.2.2 Cost Equations ... 131
3.2.3 M/G/1 Queue ... 133
3.3 Age and Residual Life* ... 136
3.3.1 Discrete Case ... 137
3.3.2 General Case ... 139
3.4 Chapter Summary ... 141
3.5 Exercises ... 142

4 Continuous Time Markov Chains ... 147
4.1 Definitions and Examples ... 147
4.2 Computing the Transition Probability ... 152
4.2.1 Branching Processes ... 157
4.3 Limiting Behavior ... 162
4.3.1 Detailed Balance Condition ... 167
4.4 Exit Distributions and Exit Times ... 170
4.4.1 Exit Distribution ... 170
4.4.2 Exit Times ... 173
4.5 Markovian Queues ... 176
4.5.1 Single Server Queues ... 177
4.5.2 Multiple Servers ... 180
4.5.3 Departure Processes ... 182
4.6 Queueing Networks* ... 183
4.7 Chapter Summary ... 190
4.8 Exercises ... 192

5 Martingales ... 201
5.1 Conditional Expectation ... 201
5.2 Examples ... 204
5.3 Gambling Strategies, Stopping Times ... 207
5.4 Applications ... 211
5.4.1 Exit Distributions ... 212
5.4.2 Exit Times ... 214
5.4.3 Extinction and Ruin Probabilities ... 216
5.4.4 Positive Recurrence of the $GI/G/1$ Queue* ... 218
5.5 Exercises ... 220

6 Mathematical Finance ... 223
6.1 Two Simple Examples ... 223
6.2 Binomial Model ... 227
6.2.1 One Period Case ... 227
6.2.2 N Period Model ... 229
6.3 Concrete Examples ... 232

6.4 American Options 237
6.5 Black–Scholes Formula 241
6.5.1 The Black–Scholes Partial Differential Equation 244
6.6 Calls and Puts 245
6.7 Exercises 247

A Review of Probability 251
A.1 Probabilities, Independence 251
A.1.1 Conditional Probability 253
A.2 Random Variables, Distributions 255
A.3 Expected Value, Moments 261
A.4 Integration to the Limit 267

References 269

Index 271