

# Contents

	<i>Foreword, by Henk Barendregt</i>	page	xiii
	<i>Preface</i>		xv
	<i>Acknowledgements</i>		xxvii
	<i>Greek alphabet</i>		xxviii
<b>1</b>	<b>Untyped lambda calculus</b>		1
1.1	Input-output behaviour of functions		1
1.2	The essence of functions		2
1.3	Lambda-terms		4
1.4	Free and bound variables		8
1.5	Alpha conversion		9
1.6	Substitution		11
1.7	Lambda-terms modulo $\alpha$ -equivalence		14
1.8	Beta reduction		16
1.9	Normal forms and confluence		19
1.10	Fixed Point Theorem		24
1.11	Conclusions		26
1.12	Further reading		27
	Exercises		29
<b>2</b>	<b>Simply typed lambda calculus</b>		33
2.1	Adding types		33
2.2	Simple types		34
2.3	Church-typing and Curry-typing		36
2.4	Derivation rules for Church's $\lambda \rightarrow$		39
2.5	Different formats for a derivation in $\lambda \rightarrow$		44
2.6	Kinds of problems to be solved in type theory		46
2.7	Well-typedness in $\lambda \rightarrow$		47
2.8	Type Checking in $\lambda \rightarrow$		50
2.9	Term Finding in $\lambda \rightarrow$		51



2.10	General properties of $\lambda \rightarrow$	53
2.11	Reduction and $\lambda \rightarrow$	59
2.12	Consequences	63
2.13	Conclusions	64
2.14	Further reading	65
	Exercises	66
<b>3</b>	<b>Second order typed lambda calculus</b>	69
3.1	Type-abstraction and type-application	69
3.2	$\Pi$ -types	71
3.3	Second order abstraction and application rules	72
3.4	The system $\lambda 2$	73
3.5	Example of a derivation in $\lambda 2$	76
3.6	Properties of $\lambda 2$	78
3.7	Conclusions	80
3.8	Further reading	80
	Exercises	82
<b>4</b>	<b>Types dependent on types</b>	85
4.1	Type constructors	85
4.2	Sort-rule and var-rule in $\lambda \omega$	88
4.3	The weakening rule in $\lambda \omega$	90
4.4	The formation rule in $\lambda \omega$	93
4.5	Application and abstraction rules in $\lambda \omega$	94
4.6	Shortened derivations	95
4.7	The conversion rule	97
4.8	Properties of $\lambda \omega$	99
4.9	Conclusions	100
4.10	Further reading	100
	Exercises	101
<b>5</b>	<b>Types dependent on terms</b>	103
5.1	The missing extension	103
5.2	Derivation rules of $\lambda P$	105
5.3	An example derivation in $\lambda P$	107
5.4	Minimal predicate logic in $\lambda P$	109
5.5	Example of a logical derivation in $\lambda P$	115
5.6	Conclusions	118
5.7	Further reading	119
	Exercises	121
<b>6</b>	<b>The Calculus of Constructions</b>	123
6.1	The system $\lambda C$	123
6.2	The $\lambda$ -cube	125



6.3	Properties of $\lambda C$	128
6.4	Conclusions	132
6.5	Further reading	133
	Exercises	134
<b>7</b>	<b>The encoding of logical notions in <math>\lambda C</math></b>	137
7.1	Absurdity and negation in type theory	137
7.2	Conjunction and disjunction in type theory	139
7.3	An example of propositional logic in $\lambda C$	144
7.4	Classical logic in $\lambda C$	146
7.5	Predicate logic in $\lambda C$	150
7.6	An example of predicate logic in $\lambda C$	154
7.7	Conclusions	157
7.8	Further reading	159
	Exercises	162
<b>8</b>	<b>Definitions</b>	165
8.1	The nature of definitions	165
8.2	Inductive and recursive definitions	167
8.3	The format of definitions	168
8.4	Instantiations of definitions	170
8.5	A formal format for definitions	172
8.6	Definitions depending on assumptions	174
8.7	Giving names to proofs	175
8.8	A general proof and a specialised version	178
8.9	Mathematical statements as formal definitions	180
8.10	Conclusions	182
8.11	Further reading	183
	Exercises	185
<b>9</b>	<b>Extension of <math>\lambda C</math> with definitions</b>	189
9.1	Extension of $\lambda C$ to the system $\lambda D_0$	189
9.2	Judgements extended with definitions	190
9.3	The rule for adding a definition	192
9.4	The rule for instantiating a definition	193
9.5	Definition unfolding and $\delta$ -conversion	197
9.6	Examples of $\delta$ -conversion	200
9.7	The conversion rule extended with $\Delta$	202
9.8	The derivation rules for $\lambda D_0$	203
9.9	A closer look at the derivation rules of $\lambda D_0$	204
9.10	Conclusions	206
9.11	Further reading	207
	Exercises	208



<b>10</b>	<b>Rules and properties of <math>\lambda</math>D</b>	211
10.1	Descriptive versus primitive definitions	211
10.2	Axioms and axiomatic notions	212
10.3	Rules for primitive definitions	214
10.4	Properties of $\lambda$ D	215
10.5	Normalisation and confluence in $\lambda$ D	219
10.6	Conclusions	221
10.7	Further reading	221
	Exercises	223
<b>11</b>	<b>Flag-style natural deduction in <math>\lambda</math>D</b>	225
11.1	Formal derivations in $\lambda$ D	225
11.2	Comparing formal and flag-style $\lambda$ D	228
11.3	Conventions about flag-style proofs in $\lambda$ D	229
11.4	Introduction and elimination rules	232
11.5	Rules for constructive propositional logic	234
11.6	Examples of logical derivations in $\lambda$ D	237
11.7	Suppressing unaltered parameter lists	239
11.8	Rules for classical propositional logic	240
11.9	Alternative natural deduction rules for $\vee$	243
11.10	Rules for constructive predicate logic	246
11.11	Rules for classical predicate logic	249
11.12	Conclusions	252
11.13	Further reading	253
	Exercises	254
<b>12</b>	<b>Mathematics in <math>\lambda</math>D: a first attempt</b>	257
12.1	An example to start with	257
12.2	Equality	259
12.3	The congruence property of equality	262
12.4	Orders	264
12.5	A proof about orders	266
12.6	Unique existence	268
12.7	The descriptor $\iota$	271
12.8	Conclusions	274
12.9	Further reading	275
	Exercises	276
<b>13</b>	<b>Sets and subsets</b>	279
13.1	Dealing with subsets in $\lambda$ D	279
13.2	Basic set-theoretic notions	282
13.3	Special subsets	287
13.4	Relations	288



13.5	Maps	291
13.6	Representation of mathematical notions	295
13.7	Conclusions	296
13.8	Further reading	297
	Exercises	302
<b>14</b>	<b>Numbers and arithmetic in <math>\lambda D</math></b>	<b>305</b>
14.1	The Peano axioms for natural numbers	305
14.2	Introducing integers the axiomatic way	308
14.3	Basic properties of the ‘new’ $\mathbb{N}$	313
14.4	Integer addition	316
14.5	An example of a basic computation in $\lambda D$	320
14.6	Arithmetical laws for addition	322
14.7	Closure under addition for natural and negative numbers	324
14.8	Integer subtraction	327
14.9	The opposite of an integer	330
14.10	Inequality relations on $\mathbb{Z}$	332
14.11	Multiplication of integers	335
14.12	Divisibility	338
14.13	Irrelevance of proof	340
14.14	Conclusions	341
14.15	Further reading	343
	Exercises	344
<b>15</b>	<b>An elaborated example</b>	<b>349</b>
15.1	Formalising a proof of Bézout’s Lemma	349
15.2	Preparatory work	352
15.3	Part I of the proof of Bézout’s Lemma	354
15.4	Part II of the proof	357
15.5	Part III of the proof	360
15.6	The holes in the proof of Bézout’s Lemma	363
15.7	The Minimum Theorem for $\mathbb{Z}$	364
15.8	The Division Theorem	369
15.9	Conclusions	371
15.10	Further reading	373
	Exercises	376
<b>16</b>	<b>Further perspectives</b>	<b>379</b>
16.1	Useful applications of $\lambda D$	379
16.2	Proof assistants based on type theory	380
16.3	Future of the field	384
16.4	Conclusions	386
16.5	Further reading	387



<b>Appendix A Logic in <math>\lambda D</math></b>	391
A.1 Constructive propositional logic	391
A.2 Classical propositional logic	393
A.3 Constructive predicate logic	395
A.4 Classical predicate logic	396
<b>Appendix B Arithmetical axioms, definitions and lemmas</b>	397
<b>Appendix C Two complete example proofs in <math>\lambda D</math></b>	403
C.1 Closure under addition in $\mathbb{N}$	403
C.2 The Minimum Theorem	405
<b>Appendix D Derivation rules for <math>\lambda D</math></b>	409
References	411
Index of names	419
Index of definitions	421
Index of symbols	423
Index of subjects	425