

# Contents

<b>Preface to the First Edition</b>	<b>vii</b>
<b>Preface to the Second Edition</b>	<b>xi</b>
<b>1 The need for measure theory</b>	<b>1</b>
1.1 Various kinds of random variables . . . . .	1
1.2 The uniform distribution and non-measurable sets . . . . .	2
1.3 Exercises . . . . .	4
1.4 Section summary . . . . .	5
<b>2 Probability triples</b>	<b>7</b>
2.1 Basic definition . . . . .	7
2.2 Constructing probability triples . . . . .	8
2.3 The Extension Theorem . . . . .	10
2.4 Constructing the Uniform[0, 1] distribution . . . . .	15
2.5 Extensions of the Extension Theorem . . . . .	18
2.6 Coin tossing and other measures . . . . .	21
2.7 Exercises . . . . .	23
2.8 Section summary . . . . .	27
<b>3 Further probabilistic foundations</b>	<b>29</b>
3.1 Random variables . . . . .	29
3.2 Independence . . . . .	31
3.3 Continuity of probabilities . . . . .	33
3.4 Limit events . . . . .	34
3.5 Tail fields . . . . .	36
3.6 Exercises . . . . .	38
3.7 Section summary . . . . .	41
<b>4 Expected values</b>	<b>43</b>
4.1 Simple random variables . . . . .	43
4.2 General non-negative random variables . . . . .	45
4.3 Arbitrary random variables . . . . .	49
4.4 The integration connection . . . . .	50
4.5 Exercises . . . . .	52
4.6 Section summary . . . . .	55
<b>5 Inequalities and convergence</b>	<b>57</b>
5.1 Various inequalities . . . . .	57
5.2 Convergence of random variables . . . . .	58
5.3 Laws of large numbers . . . . .	60
5.4 Eliminating the moment conditions . . . . .	61

5.5	Exercises . . . . .	65
5.6	Section summary . . . . .	66
<b>6</b>	<b>Distributions of random variables</b>	<b>67</b>
6.1	Change of variable theorem . . . . .	67
6.2	Examples of distributions . . . . .	69
6.3	Exercises . . . . .	71
6.4	Section summary . . . . .	72
<b>7</b>	<b>Stochastic processes and gambling games</b>	<b>73</b>
7.1	A first existence theorem . . . . .	73
7.2	Gambling and gambler's ruin . . . . .	75
7.3	Gambling policies . . . . .	77
7.4	Exercises . . . . .	80
7.5	Section summary . . . . .	81
<b>8</b>	<b>Discrete Markov chains</b>	<b>83</b>
8.1	A Markov chain existence theorem . . . . .	85
8.2	Transience, recurrence, and irreducibility . . . . .	86
8.3	Stationary distributions and convergence . . . . .	89
8.4	Existence of stationary distributions . . . . .	94
8.5	Exercises . . . . .	98
8.6	Section summary . . . . .	101
<b>9</b>	<b>More probability theorems</b>	<b>103</b>
9.1	Limit theorems . . . . .	103
9.2	Differentiation of expectation . . . . .	106
9.3	Moment generating functions and large deviations . . . . .	107
9.4	Fubini's Theorem and convolution . . . . .	110
9.5	Exercises . . . . .	113
9.6	Section summary . . . . .	115
<b>10</b>	<b>Weak convergence</b>	<b>117</b>
10.1	Equivalences of weak convergence . . . . .	117
10.2	Connections to other convergence . . . . .	119
10.3	Exercises . . . . .	121
10.4	Section summary . . . . .	122
<b>11</b>	<b>Characteristic functions</b>	<b>125</b>
11.1	The continuity theorem . . . . .	126
11.2	The Central Limit Theorem . . . . .	133
11.3	Generalisations of the Central Limit Theorem . . . . .	135
11.4	Method of moments . . . . .	137
11.5	Exercises . . . . .	139

11.6	Section summary . . . . .	142
<b>12</b>	<b>Decomposition of probability laws</b>	<b>143</b>
12.1	Lebesgue and Hahn decompositions . . . . .	143
12.2	Decomposition with general measures . . . . .	147
12.3	Exercises . . . . .	148
12.4	Section summary . . . . .	149
<b>13</b>	<b>Conditional probability and expectation</b>	<b>151</b>
13.1	Conditioning on a random variable . . . . .	151
13.2	Conditioning on a sub- $\sigma$ -algebra . . . . .	155
13.3	Conditional variance . . . . .	157
13.4	Exercises . . . . .	158
13.5	Section summary . . . . .	160
<b>14</b>	<b>Martingales</b>	<b>161</b>
14.1	Stopping times . . . . .	162
14.2	Martingale convergence . . . . .	168
14.3	Maximal inequality . . . . .	171
14.4	Exercises . . . . .	173
14.5	Section summary . . . . .	176
<b>15</b>	<b>General stochastic processes</b>	<b>177</b>
15.1	Kolmogorov Existence Theorem . . . . .	177
15.2	Markov chains on general state spaces . . . . .	179
15.3	Continuous-time Markov processes . . . . .	182
15.4	Brownian motion as a limit . . . . .	186
15.5	Existence of Brownian motion . . . . .	188
15.6	Diffusions and stochastic integrals . . . . .	190
15.7	Itô's Lemma . . . . .	193
15.8	The Black-Scholes equation . . . . .	194
15.9	Section summary . . . . .	197
<b>A</b>	<b>Mathematical Background</b>	<b>199</b>
A.1	Sets and functions . . . . .	199
A.2	Countable sets . . . . .	200
A.3	Epsilons and Limits . . . . .	202
A.4	Infimums and supremums . . . . .	204
A.5	Equivalence relations . . . . .	207

<b>B Bibliography</b>	<b>209</b>
B.1 Background in real analysis . . . . .	209
B.2 Undergraduate-level probability . . . . .	209
B.3 Graduate-level probability . . . . .	210
B.4 Pure measure theory . . . . .	210
B.5 Stochastic processes . . . . .	210
B.6 Mathematical finance . . . . .	211
<b>Index</b>	<b>213</b>