

CHAPTER		PAGE
INTRODUCTION: THE NATURE OF PROBABILITY THEORY . . . . .		1
1.	The Background . . . . .	1
2.	Procedure . . . . .	3
3.	“Statistical” Probability . . . . .	4
4.	Summary . . . . .	5
5.	Historical Note . . . . .	6
<b>I</b>	<b>THE SAMPLE SPACE . . . . .</b>	<b>7</b>
1.	The Empirical Background . . . . .	7
2.	Examples . . . . .	9
3.	The Sample Space. Events . . . . .	13
4.	Relations among Events . . . . .	14
5.	Discrete Sample Spaces . . . . .	17
6.	Probabilities in Discrete Sample Spaces: Preparations	19
7.	The Basic Definitions and Rules . . . . .	22
8.	Problems for Solution . . . . .	24
<b>II</b>	<b>ELEMENTS OF COMBINATORIAL ANALYSIS . . . . .</b>	<b>26</b>
1.	Preliminaries . . . . .	26
2.	Ordered Samples . . . . .	28
3.	Examples . . . . .	31
4.	Subpopulations and Partitions . . . . .	34
*5.	Application to Occupancy Problems . . . . .	38
*5a.	Bose-Einstein and Fermi-Dirac Statistics . . . . .	40
*5b.	Application to Runs . . . . .	42
6.	The Hypergeometric Distribution . . . . .	43
7.	Examples for Waiting Times . . . . .	47
8.	Binomial Coefficients . . . . .	50
9.	Stirling’s Formula . . . . .	52
Problems for Solution: . . . . .		54
10.	Exercises and Examples . . . . .	54

---

\* Starred sections are not required for the understanding of the sequel and should be omitted at first reading.

CHAPTER		PAGE
	11. Problems and Complements of a Theoretical Character . . . . .	58
	12. Problems and Identities Involving Binomial Coefficients . . . . .	63
*III	FLUCTUATIONS IN COIN TOSSING AND RANDOM WALKS . . . . .	67
	1. General Orientation. The Reflection Principle . . . . .	68
	2. Random Walks: Basic Notions and Notations . . . . .	73
	3. The Main Lemma . . . . .	76
	4. Last Visits and Long Leads. . . . .	78
	*5. Changes of Sign . . . . .	84
	6. An Experimental Illustration . . . . .	86
	7. Maxima and First Passages . . . . .	88
	8. Duality. Position of Maxima . . . . .	91
	9. An Equidistribution Theorem . . . . .	94
	10. Problems for Solution . . . . .	95
*IV	COMBINATION OF EVENTS . . . . .	98
	1. Union of Events . . . . .	98
	2. Application to the Classical Occupancy Problem . . . . .	101
	3. The Realization of $m$ among $N$ events . . . . .	106
	4. Application to Matching and Guessing. . . . .	107
	5. Miscellany . . . . .	109
	6. Problems for Solution . . . . .	111
V	CONDITIONAL PROBABILITY. STOCHASTIC INDEPENDENCE . . . . .	114
	1. Conditional Probability . . . . .	114
	2. Probabilities Defined by Conditional Probabilities. Urn Models . . . . .	118
	3. Stochastic Independence . . . . .	125
	4. Product Spaces. Independent Trials . . . . .	128
	*5. Applications to Genetics . . . . .	132
	*6. Sex-Linked Characters . . . . .	136
	*7. Selection . . . . .	139
	8. Problems for Solution . . . . .	140
VI	THE BINOMIAL AND THE POISSON DISTRIBUTIONS . . . . .	146
	1. Bernoulli Trials . . . . .	146
	2. The Binomial Distribution . . . . .	147
	3. The Central Term and the Tails . . . . .	150
	4. The Law of Large Numbers . . . . .	152

CHAPTER	PAGE
5. The Poisson Approximation . . . . .	153
6. The Poisson Distribution . . . . .	156
7. Observations Fitting the Poisson Distribution . . . . .	159
8. Waiting Times. The Negative Binomial Distribution . . . . .	164
9. The Multinomial Distribution . . . . .	167
10. Problems for Solution . . . . .	169
<b>VII THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION . . . . .</b>	<b>174</b>
1. The Normal Distribution . . . . .	174
2. Orientation: Symmetric Distributions . . . . .	179
3. The DeMoivre-Laplace Limit Theorem . . . . .	182
4. Examples . . . . .	187
5. Relation to the Poisson Approximation . . . . .	190
*6. Large Deviations . . . . .	192
7. Problems for Solution . . . . .	193
<b>*VIII UNLIMITED SEQUENCES OF BERNOULLI TRIALS . . . . .</b>	<b>196</b>
1. Infinite Sequences of Trials . . . . .	196
2. Systems of Gambling . . . . .	198
3. The Borel-Cantelli Lemmas . . . . .	200
4. The Strong Law of Large Numbers . . . . .	202
5. The Law of the Iterated Logarithm . . . . .	204
6. Interpretation in Number Theory Language . . . . .	208
7. Problems for Solution . . . . .	210
<b>IX RANDOM VARIABLES; EXPECTATION . . . . .</b>	<b>212</b>
1. Random Variables . . . . .	212
2. Expectations . . . . .	220
3. Examples and Applications . . . . .	223
4. The Variance . . . . .	227
5. Covariance; Variance of a Sum . . . . .	229
6. Chebyshev's Inequality . . . . .	233
*7. Kolmogorov's Inequality . . . . .	234
*8. The Correlation Coefficient . . . . .	236
9. Problems for Solution . . . . .	237
<b>X LAWS OF LARGE NUMBERS . . . . .</b>	<b>243</b>
1. Identically Distributed Variables . . . . .	243
*2. Proof of the Law of Large Numbers . . . . .	246
3. The Theory of "Fair" Games . . . . .	248

CHAPTER	PAGE
*4. The Petersburg Game . . . . .	251
5. Variable Distributions . . . . .	253
*6. Applications to Combinatorial Analysis . . . . .	256
*7. The Strong Law of Large Numbers . . . . .	258
8. Problems for Solution . . . . .	261
 XI INTEGRAL VALUED VARIABLES. GENERATING FUNCTIONS . . . . .	264
1. Generalities . . . . .	264
2. Convolutions . . . . .	266
3. Equalizations and Waiting Times in Bernoulli Trials	270
4. Partial Fraction Expansions . . . . .	275
5. Bivariate Generating Functions . . . . .	279
*6. The Continuity Theorem . . . . .	280
7. Problems for Solution . . . . .	283
 *XII COMPOUND DISTRIBUTIONS. BRANCHING PROCESSES . . . . .	286
1. Sums of a Random Number of Variables . . . . .	286
2. The Compound Poisson Distribution . . . . .	288
2a. Processes with Independent Increments . . . . .	292
3. Examples for Branching Processes . . . . .	293
4. Extinction Probabilities in Branching Processes . . . . .	295
5. The Total Progeny in Branching Processes . . . . .	298
6. Problems for Solution . . . . .	301
 XIII RECURRENT EVENTS. RENEWAL THEORY . . . . .	303
1. Informal Preparations and Examples . . . . .	303
2. Definitions . . . . .	307
3. The Basic Relations . . . . .	311
4. Examples . . . . .	313
5. Delayed Recurrent Events. A General Limit Theorem	316
6. The Number of Occurrences of $\mathcal{E}$ . . . . .	320
*7. Application to the Theory of Success Runs . . . . .	322
*8. More General Patterns . . . . .	326
9. Lack of Memory of Geometric Waiting Times . . . . .	328
10. Renewal Theory . . . . .	329
*11. Proof of the Basic Limit Theorem . . . . .	335
12. Problems for Solution . . . . .	338
 XIV RANDOM WALK AND RUIN PROBLEMS . . . . .	342
1. General Orientation . . . . .	342
2. The Classical Ruin Problem . . . . .	344

CHAPTER	PAGE
3. Expected Duration of the Game . . . . .	348
*4. Generating Functions for the Duration of the Game and for the First-Passage Times . . . . .	349
*5. Explicit Expressions . . . . .	352
6. Connection with Diffusion Processes . . . . .	354
*7. Random Walks in the Plane and Space . . . . .	359
8. The Generalized One-Dimensional Random Walk (Sequential Sampling) . . . . .	363
9. Problems for Solution . . . . .	367
 <b>XV MARKOV CHAINS . . . . .</b>	 372
1. Definition . . . . .	372
2. Illustrative Examples . . . . .	375
3. Higher Transition Probabilities . . . . .	382
4. Closures and Closed Sets . . . . .	384
5. Classification of States . . . . .	387
6. Irreducible Chains. Decompositions . . . . .	390
7. Invariant Distributions . . . . .	392
8. Transient Chains . . . . .	399
9. Periodic Chains . . . . .	404
10. Application to Card Shuffling . . . . .	406
*11. Invariant Measures. Ratio Limit Theorems . . . . .	407
*12. Reversed Chains. Boundaries . . . . .	414
13. The General Markov Process . . . . .	419
14. Problems for Solution . . . . .	424
 <b>*XVI ALGEBRAIC TREATMENT OF FINITE MARKOV CHAINS . . . . .</b>	 428
1. General Theory . . . . .	428
2. Examples . . . . .	432
3. Random Walk with Reflecting Barriers . . . . .	436
4. Transient States; Absorption Probabilities . . . . .	438
5. Application to Recurrence Times . . . . .	443
 <b>XVII THE SIMPLEST TIME-DEPENDENT STOCHASTIC PROCESSES . . . . .</b>	 444
1. General Orientation. Markov Processes . . . . .	444
2. The Poisson Process . . . . .	446
3. The Pure Birth Process . . . . .	448
*4. Divergent Birth Processes . . . . .	451
5. The Birth and Death Process . . . . .	454
6. Exponential Holding Times . . . . .	458

## CONTENTS

CHAPTER	PAGE
7. Waiting Line and Servicing Problems . . . . .	460
8. The Backward (Retrospective) Equations . . . . .	468
9. General Processes . . . . .	470
10. Problems for Solution . . . . .	478
ANSWERS TO PROBLEMS . . . . .	483
INDEX . . . . .	499