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## 1.1 What are functional data?

Figure 1.1 provides a prototype for the type of data that we shall consider. It shows the heights of 10 girls measured at a set of 31 ages in the Berkeley Growth Study (Tanner, 1981). The ages are not equally spaced: the first two are 1 and 2, followed by heights measured biennially. Although great care was taken in the measurement process, there is still a certain amount of noise in height values that has a standard deviation of about three millimeters. Even though each record involves only discrete values, these values reflect a smooth variation in height that could be as well thought of as a function of age, and is therefore a height function. Thus, the data consist of a sample of 10 functional observations  $\{h_i(t)\}$ .

There are features in this data too subtle to see in this type of plot. Figure 1.2 displays the acceleration curves  $D^2\text{Height}$ , estimated from these data by Ramsay, Hook and Gasser (1985) using a technique discussed in Chapter 5. We use the notation  $D^2$  for differentiation, as in

$$D^2\text{Height} = \frac{d^2\text{Height}}{dt^2}.$$

In Figure 1.2 the subertal growth spurt shows up as a pulse of strong positive acceleration followed by sharp negative deceleration. But most curves also show a bump at around six years that is termed the mid-spurt. We therefore conclude that some of the variation from curve to curve can be explained at the level of certain derivatives. The fact that derivatives