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The terms equidistal, equiareal, conformal, geodesic, loxodromic, cocircular, and harmonic represent examples for such classifications.

In terms of the geometry of surfaces, this is taking reference to its *first fundamental form*, namely the *Gaussian differential invariant*. In particular, in order to derive certain invariant measures of such mappings outlined in the frontline examples and called *deformation measures*, a "canonical formalism" is applied. The simultaneous diagonalization of two symmetric matrices here is of focal interest. Such a diagonalization rests on the following Theorem 1.1.

If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $B \in \mathbb{R}^{n \times n}$ is a symmetric positive-definite matrix such that the product AB^{-1} exists, then there exists a non-singular matrix X such that both following matrices are diagonal matrices, where I_n is the n -dimensional unit matrix:

$$X^T A X = \text{diag}(\lambda_1, \dots, \lambda_n), \quad X^T B X = I_n = \text{diag}(1, \dots, 1). \tag{1.1}$$

According to our understanding, the theorem had been intuitively applied by C. F. Gauss when he developed his *theory of curvature* of parameterized surfaces (two-dimensional Riemann manifold). Here, the *second fundamental form* (Hesse matrix of second derivatives, symmetric matrix H) had been analyzed with respect to the first fundamental form (a product of Jacobi matrices of first derivatives, a symmetric and positive-definite matrix G). Equivalent to the simultaneous diagonalization of a symmetric matrix H and a symmetric and positive-definite matrix G is the *general eigenvalue problem*

$$|H - \lambda G| = 0, \tag{1.2}$$

which corresponds to the *special eigenvalue problem*

$$|HG^{-1} - \lambda I_n| = 0, \tag{1.3}$$

where HG^{-1} defines the *Gaussian curvature matrix*,

$$-K = HG^{-1} \tag{1.4}$$