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McCormick (2004)

Much of present neuroscience research concerns voltage- and second-messenger-gated currents in individual cells, with the goal of understanding the cell's intrinsic neurocomputational properties. It is widely accepted that knowing the currents suffices to determine what the cell is doing and why it is doing it. This, however, contradicts a half-century-old observation that cells having identical currents can nevertheless exhibit quite different dynamics. Indeed, Hodgkin (1948) reported that some preparations could exhibit repetitive spiking with arbitrarily low frequencies, while the others discharged in a narrow frequency band. This observation was largely ignored by the neuroscience community until the seminal paper by Rinzel and Ermentrout (1989), who showed that the difference in behavior is due to different bifurcation mechanisms of excitability.

Let us treat the amplitude of the injected current in Hodgkin's experiments as a bifurcation parameter. When the amplitude is small, the cell is quiescent; when the amplitude is large, the cell fires repetitive spikes. When we change the amplitude of the injected current, the cell undergoes a transition from quiescence to repetitive spiking. From the dynamical systems point of view, the transition corresponds to a bifurcation from equilibrium to a limit cycle attractor. The type of bifurcation determines the most fundamental computational properties of neurons, such as the class of excitability, the presence or nonexistence of threshold, all-or-none spikes, subthreshold oscillations, the ability to generate postinhibitory rebound spikes, bistability of resting and spiking states, whether the neuron is an integrator or a resonator, and so on.

This book is devoted to a systematic study of the relationship between electrophysiology, bifurcations, and computational properties of neurons. The reader will learn why cells having nearly identical currents may undergo distinct bifurcations, and hence they will have fundamentally different neurocomputational properties. (Conversely, cells