

This book presents matrix algebra in a way that is well suited for those with an interest in statistics or a related discipline. It provides thorough and unified coverage of the fundamental concepts along with the specialized topics encountered in areas of statistics, such as linear statistical models and multivariate analysis. It includes a number of very useful results that have only been available from relatively obscure sources. Detailed proofs are provided for all results.

The style and level of presentation are designed to make the contents accessible to a broad audience. The book is essentially self-contained, though it is best suited for a reader who has had some previous exposure to matrices (of the kind that might be acquired in a beginning course on linear or matrix algebra). It includes exercises, it can serve as the primary text for a course on matrices or as a supplementary text in courses on such topics as linear statistical models or multivariate analysis, and it will be a valuable reference.

David A. Harville is a research staff member in the Mathematical Sciences Department of the IBM T.J. Watson Research Center. Prior to joining the Research Center, he served ten years as a mathematical statistician in the Applied Mathematics Research Laboratory of the Aerospace Research Laboratories at Wright Patterson Air Force Base, Ohio, followed by twenty years as a full professor in the Department of Statistics at Iowa State University. He has extensive experience in the area of linear statistical models, having taught (on numerous occasions) M.S.- and Ph.D.-level courses on that topic, been the thesis adviser of ten Ph.D. students, and authored over 60 research articles.

His work has been recognized by his election as a Fellow of the American Statistical Association and the Institute of Mathematical Statistics and as a member of the International Statistical Institute, and by his having served as an associate editor of *Biometrics* and of the *Journal of the American Statistical Association*.

ISBN 0-387-94978-X



ISBN 0-387-94978-X

Preface

v

1 Matrices

1

- 1.1 Basic Terminology 1
- 1.2 Basic Operations 2
- 1.3 Some Basic Types of Matrices 6
- Exercises 10

2 Submatrices and Partitioned Matrices

13

- 2.1 Some Terminology and Basic Results 13
- 2.2 Scalar Multiples, Transposes, Sums, and Products of Partitioned Matrices 17
- 2.3 Some Results on the Product of a Matrix and a Column Vector 19
- 2.4 Expansion of a Matrix in Terms of Its Rows, Columns, or Elements 20
- Exercises 22

3 Linear Dependence and Independence

23

- 3.1 Definitions 23
- 3.2 Some Basic Results 24
- Exercises 26

4 Linear Spaces: Row and Column Spaces

27

- 4.1 Some Definitions, Notation, and Basic Relationships and Properties 27

4.2	Subspaces	29
4.3	Bases	31
4.4	Rank of a Matrix	36
4.5	Some Basic Results on Partitioned Matrices and on Sums of Matrices	40
	Exercises	46
5	Trace of a (Square) Matrix	49
5.1	Definition and Basic Properties	49
5.2	Trace of a Product	50
5.3	Some Equivalent Conditions	52
	Exercises	53
6	Geometrical Considerations	55
6.1	Definitions: Norm, Distance, Angle, Inner Product, and Orthogonality	55
6.2	Orthogonal and Orthonormal Sets	61
6.3	Schwarz Inequality	62
6.4	Orthonormal Bases	63
	Exercises	68
7	Linear Systems: Consistency and Compatibility	71
7.1	Some Basic Terminology	71
7.2	Consistency	72
7.3	Compatibility	73
7.4	Linear Systems of the Form $A'AX = A'B$	74
	Exercise	77
8	Inverse Matrices	79
8.1	Some Definitions and Basic Results	79
8.2	Properties of Inverse Matrices	81
8.3	Premultiplication or Postmultiplication by a Matrix of Full Column or Row Rank	83
8.4	Orthogonal Matrices	84
8.5	Some Basic Results on the Ranks and Inverses of Partitioned Matrices	88
	Exercises	103
9	Generalized Inverses	107
9.1	Definition, Existence, and a Connection to the Solution of Linear Systems	107
9.2	Some Alternative Characterizations	109
9.3	Some Elementary Properties	117
9.4	Invariance to the Choice of a Generalized Inverse	119
9.5	A Necessary and Sufficient Condition for the Consistency of a Linear System	120

9.6	Some Results on the Ranks and Generalized Inverses of Partitioned Matrices	121
9.7	Extension of Some Results on Systems of the Form $\mathbf{AX} = \mathbf{B}$ to Systems of the Form $\mathbf{AXC} = \mathbf{B}$	125
	Exercises	126
10	Idempotent Matrices	133
10.1	Definition and Some Basic Properties	133
10.2	Some Basic Results	134
	Exercises	136
11	Linear Systems: Solutions	139
11.1	Some Terminology, Notation, and Basic Results	139
11.2	General Form of a Solution	140
11.3	Number of Solutions	142
11.4	A Basic Result on Null Spaces	143
11.5	An Alternative Expression for the General Form of a Solution	144
11.6	Equivalent Linear Systems	145
11.7	Null and Column Spaces of Idempotent Matrices	146
11.8	Linear Systems With Nonsingular Triangular or Block-Triangular Coefficient Matrices	146
11.9	A Computational Approach	149
11.10	Linear Combinations of the Unknowns	150
11.11	Absorption	152
11.12	Extensions to Systems of the Form $\mathbf{AXC} = \mathbf{B}$	157
	Exercises	158
12	Projections and Projection Matrices	161
12.1	Some General Results, Terminology, and Notation	161
12.2	Projection of a Column Vector	163
12.3	Projection Matrices	166
12.4	Least Squares Problem	169
12.5	Orthogonal Complements	171
	Exercises	175
13	Determinants	177
13.1	Some Definitions, Notation, and Special Cases	177
13.2	Some Basic Properties of Determinants	181
13.3	Partitioned Matrices, Products of Matrices, and Inverse Matrices	185
13.4	A Computational Approach	189
13.5	Cofactors	189
13.6	Vandermonde Matrices	193
13.7	Some Results on the Determinant of the Sum of Two Matrices	195
13.8	Laplace's Theorem and the Binet-Cauchy Formula	197
	Exercises	202

14 Linear, Bilinear, and Quadratic Forms	207
14.1 Some Terminology and Basic Results	207
14.2 Nonnegative Definite Quadratic Forms and Matrices	210
14.3 Decomposition of Symmetric and Symmetric Nonnegative Definite Matrices	215
14.4 Generalized Inverses of Symmetric Nonnegative Definite Matrices	220
14.5 LDU, U'DU, and Cholesky Decompositions	221
14.6 Skew-Symmetric Matrices	237
14.7 Trace of a Nonnegative Definite Matrix	238
14.8 Partitioned Nonnegative Definite Matrices	240
14.9 Some Results on Determinants	245
14.10 Geometrical Considerations	252
14.11 Some Results on Ranks and Row and Column Spaces and on Linear Systems	256
14.12 Projections, Projection Matrices, and Orthogonal Complements Exercises	257 273
15 Matrix Differentiation	285
15.1 Definitions, Notation, and Other Preliminaries	286
15.2 Differentiation of (Scalar-Valued) Functions: Some Elementary Results	292
15.3 Differentiation of Linear and Quadratic Forms	294
15.4 Differentiation of Matrix Sums, Products, and Transposes (and of Matrices of Constants)	296
15.5 Differentiation of a Vector or (Unrestricted or Symmetric) Matrix With Respect to Its Elements	299
15.6 Differentiation of a Trace of a Matrix	300
15.7 The Chain Rule	302
15.8 First-Order Partial Derivatives of Determinants and Inverse and Adjoint Matrices	304
15.9 Second-Order Partial Derivatives of Determinants and Inverse Matrices	308
15.10 Differentiation of Generalized Inverses	309
15.11 Differentiation of Projection Matrices	314
15.12 Evaluation of Some Multiple Integrals	320
Exercises	323
Bibliographic and Supplementary Notes	331
16 Kronecker Products and the Vec and Vech Operators	333
16.1 The Kronecker Product of Two or More Matrices: Definition and Some Basic Properties	333
16.2 The Vec Operator: Definition and Some Basic Properties	339
16.3 Vec-Permutation Matrix	343
16.4 The Vech Operator	350

16.5	Reformulation of a Linear System	363
16.6	Some Results on Jacobian Matrices	365
	Exercises	368
	Bibliographic and Supplementary Notes	374
17	Intersections and Sums of Subspaces	377
17.1	Definitions and Some Basic Properties	377
17.2	Some Results on Row and Column Spaces and on the Ranks of Partitioned Matrices	383
17.3	Some Results on Linear Systems and on Generalized Inverses of Partitioned Matrices	390
17.4	Subspaces: Sum of Their Dimensions Versus Dimension of Their Sum	393
17.5	Some Results on the Rank of a Product of Matrices	396
17.6	Projections Along a Subspace	399
17.7	Some Further Results on the Essential Disjointness and Orthog- onality of Subspaces and on Projections and Projection Matrices Exercises	406 408
	Bibliographic and Supplementary Notes	414
18	Sums (and Differences) of Matrices	415
18.1	Some Results on Determinants	416
18.2	Some Results on Inverses and Generalized Inverses and on Linear Systems	419
18.3	Some Results on Positive (and Nonnegative) Definiteness	432
18.4	Some Results on Idempotency	434
18.5	Some Results on Ranks	440
	Exercises	445
	Bibliographic and Supplementary Notes	453
19	Minimization of a Second-Degree Polynomial (in n Variables) Subject to Linear Constraints	455
19.1	Unconstrained Minimization	456
19.2	Constrained Minimization	459
19.3	Explicit Expressions for Solutions to the Constrained Minimization Problem	464
19.4	Some Results on Generalized Inverses of Partitioned Matrices	473
19.5	Some Additional Results on the Form of Solutions to the Con- strained Minimization Problem	479
19.6	Transformation of the Constrained Minimization Problem to an Unconstrained Minimization Problem	485
19.7	The Effect of Constraints on the Generalized Least Squares Problem	486
	Exercises	488
	Bibliographic and Supplementary Notes	491

20	The Moore-Penrose Inverse	493
20.1	Definition, Existence, and Uniqueness (of the Moore-Penrose Inverse)	493
20.2	Some Special Cases	495
20.3	Special Types of Generalized Inverses	496
20.4	Some Alternative Representations and Characterizations	502
20.5	Some Basic Properties and Relationships	504
20.6	Minimum Norm Solution to the Least Squares Problem	507
20.7	Expression of the Moore-Penrose Inverse as a Limit	508
20.8	Differentiation of the Moore-Penrose Inverse	510
	Exercises	513
	Bibliographic and Supplementary Notes	514
21	Eigenvalues and Eigenvectors	515
21.1	Definitions, Terminology, and Some Basic Results	516
21.2	Eigenvalues of Triangular or Block-Triangular Matrices and of Diagonal or Block-Diagonal Matrices	522
21.3	Similar Matrices	524
21.4	Linear Independence of Eigenvectors	528
21.5	Diagonalization	531
21.6	Expressions for the Trace and Determinant of a Matrix	539
21.7	Some Results on the Moore-Penrose Inverse of a Symmetric Matrix	540
21.8	Eigenvalues of Orthogonal, Idempotent, and Nonnegative Definite Matrices	541
21.9	Square Root of a Symmetric Nonnegative Definite Matrix	543
21.10	Some Relationships	545
21.11	Eigenvalues and Eigenvectors of Kronecker Products of (Square) Matrices	547
21.12	Singular Value Decomposition	550
21.13	Simultaneous Diagonalization	559
21.14	Generalized Eigenvalue Problem	562
21.15	Differentiation of Eigenvalues and Eigenvectors	564
21.16	An Equivalence (Involving Determinants and Polynomials)	567
	Appendix: Some Properties of Polynomials (in a Single Variable)	573
	Exercises	575
	Bibliographic and Supplementary Notes	581
22	Linear Transformations	583
22.1	Some Definitions, Terminology, and Basic Results	583
22.2	Scalar Multiples, Sums, and Products of Linear Transformations	589
22.3	Inverse Transformations and Isomorphic Linear Spaces	592
22.4	Matrix Representation of a Linear Transformation	595
22.5	Terminology and Properties Shared by a Linear Transformation and Its Matrix Representation	603

22.6	Linear Functionals and Dual Transformations	606
	Exercises	609
	References	615
	Index	619