

Contents

1	Preface	13
2	Distribution functions in one-dimensional case	17
2.1	Basic notations and properties of d.f.s	17
2.1.1	Riemann-Stieltjes integral	22
2.2	Basic properties of the set $G(x_n)$ of all d.f.s of x_n	23
2.3	Continuity points of $g(x) \in G(x_n)$	29
2.4	Lower and upper d.f. of x_n	31
2.5	Everywhere dense sequence x_n in $[0, 1]$	33
2.6	Singleton $G(x_n) = \{g(x)\}$	33
2.7	Uniform distribution (u.d.) of x_n	42
2.7.1	Examples of u.d. sequences	47
3	Examples of applications of d.f.s	49
3.1	(λ, λ') -distribution	49
3.2	Statistically independent sequences	50
3.3	Statistical limit points	55
3.3.1	Examples	56
3.4	Statistically convergent sequences	58
3.4.1	Examples	60
3.5	Diophantine approximation generalized	63
3.6	The classical Diophantine approximation	66
3.7	Uniformly maldistributed sequences	67
3.8	The sequence $\xi(3/2)^n \bmod 1$	68
3.9	Benford's law	74
3.9.1	Basic results	75
3.9.2	General scheme of a solution of the First Digit Problem	78
3.10	D.f.s of a two dimensional sequence (x_n, y_n) , both x_n and y_n are u.d. (copulas)	84
3.11	Ratio block sequences	84
4	Calculation of d.f.s	87
4.1	$G(f(n) \bmod 1)$ directly from the definition of d.f.s	87
4.2	Proof of $G(x_n) = H$ using the connectivity of $G(x_n)$	97
4.3	Proof of $G(x_n) \subset H$ using L^2 discrepancy	97
4.4	Proof of $G(x_n) \subset H$ solving $\int_0^1 \int_0^1 F(x, y) dg(x) dg(y) = 0$	98

4.5	Linear combination $\{tg_1(x) + (1-t)g_2(x); t \in [0, 1]\}$	103
4.6	Other characterization of $\{tg_1(x) + (1-t)g_2(x); t \in [0, 1]\}$	105
4.7	Computing $G(h(x_n, y_n))$ by $\int_{\{h(x,y)\} < t} 1.dg(x, y)$	110
4.7.1	Computation of $G((\log n, \log \log n) \bmod 1)$	113
4.8	Computation of $G(x_n)$ using mapping $x_n \rightarrow f(x_n)$	121
4.8.1	Mapping $x_n \rightarrow f(x_n)$ and mapping $g \rightarrow g_f$	121
4.8.2	One-to-one map $g \rightarrow g_f$	123
4.8.3	Simple examples of $f : [0, 1] \rightarrow [0, 1]$ and $g_f(x)$	124
4.8.4	Solution of $g_f = g_1$	126
4.9	Hausdorff moment problem (an information)	128
4.10	The moment problem $\mathbf{X} = \mathbf{F}(g)$ for d.f. $g(x)$	130
4.10.1	The body Ω of all $\mathbf{F}(g)$ and the boundary $\partial\Omega$	131
4.10.2	Solution of $\mathbf{X} = \mathbf{F}(g)$ for $\mathbf{X} \in \Omega$ and $\mathbf{X} \in \partial\Omega$	134
4.10.3	Proof of the solutions of $\mathbf{X} = \mathbf{F}(g)$	137
4.10.4	The basic property of Ω	138
4.10.5	The law of composition $\mathbf{X} = \sum_{i=0}^N \mathbf{a}_i + \mathbf{B}_i \mathbf{X}^{(i)}$	142
4.10.6	Linear neighbourhoods: Definition and construction	145
4.10.7	Criteria for $\mathbf{F}(g) \in \partial\Omega$	149
5	Operations with d.f.s	163
5.1	Statistical independence	163
5.2	Simple operations with sequences	169
5.3	Convolution of d.f.s (an information)	173
5.4	Characteristic function (an information)	175
5.5	Random variables (Some results)	176
6	Special sequences	177
6.1	D.f.s of $\xi(3/2)^n \bmod 1$ (continuation of Section 3.8)	177
6.1.1	Functional equation $g_f = g_h$	177
6.1.2	Intervals of uniqueness for $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$	178
6.1.3	$\int_0^1 \int_0^1 F(x, y) dg(x) dg(y) = 0$ characterizes $g_f = g_h$	180
6.1.4	$g(x)$ and $1 - g(1 - x)$ satisfies $g_f = g_h$ simultaneously	185
6.1.5	Explicit formula for $F(x, y) =$ $ \{2x\} - \{3y\} + \{2y\} - \{3x\} - \{2x\} - \{2y\} - \{3x\} - \{3y\} $	189
6.1.6	Integration $\int_0^1 \int_0^1 F(x, y) dg(x) dg(y)$ by parts. Technical preparation	191
6.1.7	Copositivity of $F(x, y)$	200

6.1.8	Solution of $g_f = g_h$	201
6.1.9	Examples of solutions $g_f = g_h$	206
6.1.10	Problem $g_f = g_h$ and $g(x) = 1$ for $x \in [1/2, 1]$ (Theorem 117)	208
6.1.11	Mahler's conjecture	212
6.1.12	Some results without use Theorem 117	212
6.1.13	Examples	214
6.2	Mappings g_f, g_h for $f(x) = 4x(1-x), h(x) = x(4x-3)^2$	221
6.2.1	Intervals of uniqueness for $f(x) = 4x(1-x) \bmod 1$ and $h(x) = x(4x-3)^2 \bmod 1$	221
6.3	Equations for $g(x) \in G((3/2)^n \bmod 1)$ other as $g_f = g_h$	230
6.4	Ratio block sequences (continuation of 3.11)	233
6.4.1	Basic notations of $X_n = (x_1/x_n, \dots, x_n/x_n)$	234
6.4.2	Overview of basic results of d.f.s $g(x) \in G(X_n)$	234
6.4.3	Basic theorems of $G(X_n)$	237
6.4.4	Continuity of $g \in G(X_n)$	240
6.4.5	Singleton $G(X_n) = \{g\}$	242
6.4.6	One-step d.f. $c_\alpha(x)$ of $G(X_n)$	244
6.4.7	Connectivity of $G(X_n)$	248
6.4.8	Everywhere density of $x_m/x_n, m, n = 1, 2, \dots$	255
6.4.9	U.d. of X_n	261
6.4.10	L^2 discrepancy of X_n	262
6.4.11	Boundaries of $g(x) \in G(X_n)$	263
6.4.12	Applications of boundaries of $g(x)$	267
6.4.13	Lower and upper d.f.s of $G(X_n)$	268
6.4.14	Algorithm for constructing $\tilde{g}(x) \leq g(x)$	272
6.4.15	$g(x) \in G(X_n)$ with constant intervals	275
6.4.16	Transformation of X_n by $1/x \bmod 1$	276
6.4.17	Construction $H \subset G(X_n)$ of d.f.s	277
6.4.18	Examples	281
6.4.19	Block sequence $A_n = (1/q_n, a_2/q_n, \dots, a_{\varphi(q_n)}/q_n)$ of re- duced rational numbers	304
6.4.20	Block sequence $A_n = (1/q_n, 2/q_n, \dots, q_n/q_n)$ of non- reduced rational numbers	308
6.5	The sequence $\varphi(n)/n, n \in (k, k+N]$	311
6.5.1	A necessary and sufficient condition that $\lim_{m \rightarrow \infty} F(k_m, k_m + N_m) = g_0(x)$	313
6.5.2	The Erdős's approach	316

6.5.3	Examples	324
6.5.4	x -numbers	326
6.5.5	Schinzel–Wang theorem	331
6.5.6	Additional property of $\varphi(n)/n$	334
6.5.7	Upper bound of $\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=1}^N \left \frac{\varphi(m)}{m} - \frac{\varphi(n)}{n} \right $	335
6.5.8	Lower bound of $\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=1}^N \left \frac{\varphi(m)}{m} - \frac{\varphi(n)}{n} \right $	336
6.6	Benford's law (B.L.) (continuation of 3.9, p. 74)	337
6.6.1	D.f. of sequences involving logarithm	337
6.6.2	B.L. for natural numbers n powers on r	338
6.6.3	B.L. for primes p powers on r	339
6.6.4	The same B.L. for natural and prime numbers	340
6.6.5	Rate of convergence of $F_N(x)$ of the sequence $\log_b n^r \bmod 1$	340
6.6.6	Rate of convergence of $F_N(x)$ for $\log_b p_n^r \bmod 1$ with primes p_n	345
6.6.7	Discrepancy D_N of the sequence $\log_b n^r \bmod 1$	351
6.6.8	Benford's law of x_n and properties of $G(x_n)$	352
6.6.9	Two-dimensional Benford's law	362
6.7	Gauss-Kuzmin theorem and $g(x) = g_f(x)$	365
6.8	Uniformly maldistributed sequences (continuation of 3.7, p. 67)	369
6.8.1	Applications of u.m. sequences	373
6.8.2	The multidimensional u.m.	375
7	The multi-dimensional d.f.s	381
7.1	The two-dimensional d.f.s.: basic results	381
7.2	The multidimensional d.f.s	386
7.3	L^2 discrepancies	388
7.4	Multidimensional generalization of L^2 discrepancy	392
7.5	A.d.f. of $(x_n, x_{n+k_1}, x_{n+k_2}, \dots, x_{n+k_{s-1}})$	396
7.6	Computation of integrals by d.f.s	399
8	Copulas	405
8.1	The set of s -dimensional copulas $\mathbf{G}_{s,1}$	405
8.2	Dimension $s = 2$	406
8.3	Basic properties of $G_{2,1}$	408
8.4	Dimension $s = 3$	410
8.5	Sequences (x_n, y_n) with both u.d. x_n and y_n	416

8.6	Sequences (x_n, y_n, z_n) where $(x_n, y_n), (x_n, z_n),$ and (y_n, z_n) are u.d.	422
8.7	Method of d.f.s for $\frac{1}{N} \sum_{n=1}^N F(x_n, y_n)$	423
8.8	Boundaries of $\frac{1}{N} \sum_{n=1}^N F(x_n, y_n)$	424
8.9	D.f. $g(x, y)$ with given marginal $g(1, y)$ and $g(x, 1)$	427
8.10	D.f. of $(x_n, x_{n+1}, \dots, x_{n+s-1})$ where x_n is van der Corput . . .	429
8.10.1	Two-dimensional shifted van der Corput sequence . . .	429
8.10.2	Three-dimensional shifted van der Corput sequence . .	437
8.10.3	Four-dimensional shifted van der Corput sequence . . .	443
8.10.4	s -th iteration of von Neumann-Kakutani transformation	448
9	Extremes of an integral over copulas	451
9.1	Criterion for maximality	452
9.1.1	Summarization	463
9.2	Extremes of $\int_0^1 \int_0^1 F(x, y) d_x d_y g(x, y)$ attained at shuffles of M	469
9.3	Extremes $\int_0^1 \int_0^1 F(x, y) d_x d_y g(x, y)$ for $F(x, y) = \Phi(x + y)$	471
9.4	Extremes of $\int_0^1 f_1(\Phi(x)) f_2(\Psi(x)) dx$	472
9.5	Extremes of $\int_0^1 f(x) \Phi(x) dx$	473
9.5.1	Piecewise linear $f(x)$	474
9.6	Computation of $\int_0^1 f_1(\Phi(x)) f_2(\Psi(x)) dx$	478
9.7	Extremes of $\int_0^1 \int_0^1 \int_0^1 F(x, y, z) d_x d_y d_z g(x, y, z)$	480
10	Solution of an integral equation over d.f.s	481
10.1	Examples of calculations of $\int_0^1 \int_0^1 F(x, y) dg(x) dg(y)$	481
10.2	Theorems of calculations of $\int_0^1 \int_0^1 F(x, y) dg(x) dg(y)$	485
10.3	Copositive $F(x, y)$	491
10.4	Copositive $F(x, y)$ having $F''_{xy} = 0$	496
10.5	The matrix form of $\int_0^1 \int_0^1 F(x, y) dg(x) dg(y)$	497
10.6	Examples of the matrix forms	498
11	Miscellaneous examples of d.f.s	505
11.1	Trigonometric sequences	505
11.2	Logarithmic sequences	507
11.3	Sequence involving $n\alpha \bmod 1$	508
11.4	Sequence $(\{n\alpha\}, \{(n+1)\alpha\}, \{(n+2)\alpha\})$	511
11.5	Sequences of the type $f(x_n)$	512
11.6	Sequence of a scalar product	514

11.7	β -van der Corput sequence	516
11.8	Jager's example	517
11.9	Uniformly quick sequences	518
11.10	Benford's law of binomial coefficients	520
11.11	Exponential sequences	522
11.12	A new proof of u.d. of van der Corput sequence	528
12	Different themes	535
12.1	The mean square worst-case error	535
12.2	Uniform distribution preserving map (u.d.p.)	541
12.2.1	Multidimensional u.d.p. map	543
12.3	Discrepancies of $ x_m - x_n $	544
13	Integral formulas	549
13.1	Integral over $ x - y $	549
13.2	Generalized $L^{(2)}$ discrepancies	551
13.3	Euclidean $L^{(2)}$ discrepancies	554
14	Name index	559
15	Subject index	563
16	Symbols and abbreviations	567
	References	571