Contents

1 Introductory Concepts **1**

I. BASIC BUILDING BLOCKS, 1

1. Set, 1
2. Group, 1
3. Field, 3
4. Linear Vector Space, 5
5. Algebra, 8

II. BASES, 10

1. For a Group, 11
2. For a Field, 12
3. For a Vector Space, 12
4. For an Algebra, 12

III. MAPPINGS, REALIZATIONS, REPRESENTATIONS, 13

1. Sets, 14
2. Groups, 15
3. Fields, 16
4. Vector Spaces, 18
5. Algebras, 19

RÉSUMÉ, 19
EXERCISES, 20
NOTES AND REFERENCES, 23

2 The Classical Groups **24**

I. GENERAL LINEAR GROUPS, 24

1. Change of Basis, 24
2. Covariance and Contravariance, 25

II. VOLUME PRESERVING GROUPS, 26

1. *Direct Sum*, 26
2. *Direct Product*, 28
3. *Symmetric Reduction in Tensor Space*, 30
4. *Fully Symmetric Subspaces—Expansion of Functions*, 33
5. *Fully Antisymmetric Subspaces—Volume Element*, 36

III. METRIC PRESERVING GROUPS, 37

1. *The Metric*, 37
2. *Kinds of Metrics*, 39
3. *Weyl Unitary Trick*, 42
4. *Metrics in Function Spaces*, 43
5. *Metric Preserving Groups*, 43

IV. PROPERTIES OF THE CLASSICAL GROUPS, 47

1. *Relationships Among the Classical Groups*, 47
2. *Dimensions of the Classical Groups*, 49
3. *Isomorphisms and Homomorphisms Among the Classical Groups*, 52

RÉSUMÉ, 52
EXERCISES, 53
NOTES AND REFERENCES, 56

3 Continuous Groups—Lie Groups 57

I. TOPOLOGICAL GROUPS, 57

1. *Some Basic Definitions*, 57
2. *Comments*, 65
3. *Additional Comments*, 66

II. AN EXAMPLE, 66

1. *The Two-Parameter Group of Transformations on the Straight Line R_1*, 66
2. *Some Realizations for This Group*, 69

III. ADDITIONAL NECESSARY CONCEPTS, 71

1. *Topological Concepts*, 71
2. *Algebraic Concepts*, 72
3. *Local Concepts*, 76

IV. LIE GROUPS, 77

1. *The Motivation*, 77

II. VOLUME PRESERVING GROUPS, 26

1. Direct Sum, 26
2. Direct Product, 28
3. Symmetric Reduction in Tensor Space, 30
4. Fully Symmetric Subspaces—Expansion of Functions, 33
5. Fully Antisymmetric Subspaces—Volume Element, 36

III. METRIC PRESERVING GROUPS, 37

1. The Metric, 37
2. Kinds of Metrics, 39
3. Weyl Unitary Trick, 42
4. Metrics in Function Spaces, 43
5. Metric Preserving Groups, 43

IV. PROPERTIES OF THE CLASSICAL GROUPS, 47

1. Relationships Among the Classical Groups, 47
2. Dimensions of the Classical Groups, 49
3. Isomorphisms and Homomorphisms Among the Classical Groups, 52

RÉSUMÉ, 52
EXERCISES, 53
NOTES AND REFERENCES, 56

3 Continuous Groups—Lie Groups 57

I. TOPOLOGICAL GROUPS, 57

1. Some Basic Definitions, 57
2. Comments, 65
3. Additional Comments, 66

II. AN EXAMPLE, 66

1. The Two-Parameter Group of Transformations on the Straight Line R_1, 66
2. Some Realizations for This Group, 69

III. ADDITIONAL NECESSARY CONCEPTS, 71

1. Topological Concepts, 71
2. Algebraic Concepts, 72
3. Local Concepts, 76

IV. LIE GROUPS, 77

1. The Motivation, 77

V. THE INVARIANT INTEGRAL, 78

1. The Rearrangement Property, 78
2. Reparameterization of G_0, 79
3. General Left and Right Invariant Densities, 80
4. Equality of Left and Right Measures, 81
5. Extension to Continuous Groups, 84

RÉSUMÉ, 84
EXERCISES, 84
NOTES AND REFERENCES, 86

4 Lie Groups and Lie Algebras **87**

I. INFINITESIMAL PROPERTIES OF LIE GROUPS, 87

1. Infinitesimal Generators for Lie Groups of Transformations, 87
2. Infinitesimal Generators for a Lie Group, 91
3. Infinitesimal Generators for Matrix Groups, 92
4. Commutation Relations, 94

II. LIE'S FIRST THEOREM, 96

1. Theorem, 96
2. Example, 98
3. Comment, 99

III. LIE'S SECOND THEOREM, 100

1. Theorem, 100
2. Example, 103
3. Comment, 103

IV. LIE'S THIRD THEOREM, 104

1. Theorem, 104
2. Structure Constants as Matrix Elements, 105

V. CONVERSES OF LIE'S THREE THEOREMS, 106

1. The Converses, 107
2. Comments, 108
3. Example, 109
4. An Important Comment, 109

VI. TAYLOR'S THEOREM FOR LIE GROUPS, 111

1. The Theorem, 111
2. An Auxiliary Result, 113

3. *Comment*, 114
4. *τ-Ordered Products*, 114

RÉSUMÉ, 117
EXERCISES, 118
NOTES AND REFERENCES, 119

5 Some Simple Examples **120**

I. RELATIONS AMONG SOME LIE ALGEBRAS, 121

1. *1 × 1 Quaternion Groups*, 121
2. *2 × 2 Unitary Groups*, 122
3. *3 × 3 Orthogonal Groups*, 124

II. COMPARISON OF LIE GROUPS, 126

1. *Parameter Spaces for $Sl(1, q)$, $SU(2, c)$, $SO(3, r)$*, 126
2. *Connectivity*, 129
3. *Homotopy and Discrete Invariant Subgroups*, 133

III. REPRESENTATIONS OF $SU(2, c)$, 135

1. *General Considerations*, 135
2. *Tensor Product Representations*, 136
3. *Representations of $SO(3, r)$*, 141

IV. QUATERNION COVERING GROUP, 142

1. *Direct Sums and Products*, 142
2. *Representations of the Factor Groups*, 142
3. *Two Possible Physical Consequences*, 145

V. SPIN AND DOUBLE-VALUEDNESS—DESCRIPTION OF THE ELECTRON, 146

VI. NONCANONICAL PARAMETERIZATIONS FOR $SU(2; c)$, 149

1. *Baker-Campbell-Hausdorff Formulas*, 149
2. *Application in Constructing Representations*, 153
3. *Physical Applications*, 158

RÉSUMÉ, 178
EXERCISES, 179
NOTES AND REFERENCES, 181

6 Classical Algebras **182**

I. COMPUTATION OF THE ALGEBRAS, 182

1. General Procedures, 182
2. Unitary Groups, 183
3. Orthogonal Groups, 186
4. Symplectic Groups, 187
5. Bases for These Algebras, 189
6. Origin of the Embedding Groups SO(2n) and SU*(2n)*, 192
7. Summary of the Real Forms of the Classical Groups, 197

II. TOPOLOGICAL PROPERTIES, 199

1. Connectivity, 199
2. Cosets, 200
3. Contraction, 209

RÉSUMÉ, 212
EXERCISES, 213
NOTES AND REFERENCES, 215

7 Lie Algebras and Root Spaces **216**

I. GENERAL STRUCTURE THEORY FOR LIE ALGEBRAS, 216

1. The Basic Tools, 216
2. The Regular Representation, 218
3. Systematics of Subalgebras, 220
4. Lie's Theorem, 229
5. Classification of Lie Algebras, 234

II. THE SECULAR EQUATION, 237

1. Rank, 237
2. Jordan Canonical Form, 241
3. First Criterion of Solvability, 243
4. Properties of the Root Subspaces, 244

III. THE METRIC, 248

1. Motivation for Choice, 248
2. Properties and Examples, 249
3. Metrics in Other Representations, 251
4. Second Criterion of Solvability, 253

IV. CARTAN'S CRITERION, 255

1. The Criterion, 255
2. Comments and Examples, 256
3. Complete Reducibility of Semisimple Algebras, 259

V. CANONICAL COMMUTATION RELATIONS FOR SEMISIMPLE ALGEBRAS, 261

1. Structure of the Metric in a Root Subspace Decomposition, 261
2. Properties of the Cartan Subalgebra, 263
3. First Chain Condition, 264
4. Second Chain Condition, 265
5. The Canonical Commutation Relations, 270

RÉSUMÉ, 273
EXERCISES, 274
NOTES AND REFERENCES, 277

8 Root Spaces and Dynkin Diagrams **278**

I. CLASSIFICATION OF THE SIMPLE ROOT SPACES, 278

1. Review of the Canonical Commutation Relations, 278
2. The Rank-2 Root Spaces, 280
3. Construction of the Simple Root Spaces, 285
4. The E Series, 288
5. List of the Simple Root Spaces, 290

II. IDENTIFICATION OF THE CLASSICAL ALGEBRAS, 293

1. A_n *and* $Sl(n+1; c)$, 293
2. The Sibling Algebras D_n, B_n, C_n, 294

III. DYNKIN DIAGRAMS, 298

1. Simple Roots, 298
2. Properties of the Dynkin Diagrams, 306

RÉSUMÉ, 315
EXERCISES, 316
NOTES AND REFERENCES, 317

9 Real Forms **318**

I. ALGEBRAIC MACHINERY, 319

1. Automorphisms, 319

2. *Some Relations Between Algebraic and Topological Properties*, 326
3. *The Classification Machinery*, 334

II. CLASSIFICATION OF THE REAL FORMS, 339

1. A_{n-1}, 339
2. B_n, 342
3. D_n, 342
4. C_n, 343
5. *Exceptional Groups*, 345

III. DISCUSSION OF RESULTS, 346

1. *Tables of the Real Forms*, 346
2. *The Character Function*, 348

IV. PROPERTIES OF COSETS, 350

1. *Matrix Properties*, 350
2. *Inner Products and Index*, 353
3. *Rank*, 354

V. ANALYTICAL PROPERTIES OF COSETS, 363

1. *Structure of the Classical Cosets*, 363
2. *Global Transformation Properties*, 366
3. *Infinitesimal Transformations*, 380
4. *Infinitesimal's Transformations*, 383
5. *Geodesics, Distance, Metric*, 385
6. *Measure and Volume on Cosets*, 391
7. *Measure and Volume on the Classical Groups*, 395
8. *Canonical Coset Parameterization of the Classical Groups*, 401

VI. REAL FORMS OF THE SYMMETRIC SPACES, 406

1. *Algebraic Machinery*, 406
2. *Tables of the Real Forms*, 410
3. *Characters of the Real Forms*, 421

RÉSUMÉ, 422
EXERCISES, 422
NOTES AND REFERENCES, 435

10 Contractions and Expansions 436

I. SIMPLE CONTRACTIONS, 436

1. *The Little Prince*, 436
2. *Inönü-Wigner Contractions*, 446
3. *Some Useful Contractions*, 450
4. *The Baker-Campbell-Hausdorff Formula*, 460

II. SALETAN CONTRACTIONS, 463

1. *The First Contraction*, 463
2. *Further Contractions*, 469
3. *Saletan Contractions in a Kupczynski Basis*, 472

III. EXPANSIONS, 477

1. *Examples of Useful Expansions*, 478
2. *Expansions of Rank-1 Spaces*, 483

RÉSUMÉ, 492

EXERCISES, 492

NOTES AND REFERENCES, 505

Bibliography 507

Author Index 565

Subject Index 567