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The goal of this text is two fold. First, we wish to present the reader with an introductory, but thorough, development of the mathematical aspects of optimal control theory. This is done in a "graded" way, as the most basic problem, with a continuous time ODE, is explained in Chapter 1, and increasingly more complicated problems are handled as the book progresses. This includes variations of the initial conditions, imposed bounds on the control, multiple states and controls, linear dependence on the control, and free terminal time. Optimal control of discrete systems and optimal control of partial differential equations are also introduced.

The second goal is to give the reader an insight into application of optimal control theory to biological models. Several different kinds of applications are presented here, including disease models of immunology and epidemic types, management decisions in harvesting and resource allocation models, and more. These are presented in the interactive "lab" sections, which we feel is a novel feature of this text. The MATLAB codes on which the labs are based are included, in addition to a user-friendly interface, which will allow everyone, even those with no prior MATLAB knowledge, to access them. The underlying numerical methods are also developed in the text.

This book is designed for use as a textbook for advanced undergraduate or beginning graduate students. It would be suitable for a one-semester course. It can also be used by anyone who wants to learn optimal control theory for application to specific models. Mathematically, only a basic knowledge of multi-variable calculus and simple ordinary differential equations is needed for the bulk of the text. Some prior knowledge of PDEs is required for the (optional) chapter on this subject. The reader should also be familiar with mathematical models and how they are used. This book is not intended as a course in mathematical modeling.