CONTENTS

PREFACE TO THE REVISED 2010 EDITION iii

PREFACE v

I AXIOMATIC SET THEORY 1

CHAPTER 1 GENERAL BACKGROUND 3
§1 What is infinity? 3
§2 Countable or uncountable? 4
§3 A non-denumerable set 6
§4 Larger and smaller 7
§5 The continuum problem 9
§6 Significance of the results 9
§7 Frege set theory 11
§8 Russell's paradox 11
§9 Zermelo set theory 12
§10 Sets and classes 13

CHAPTER 2 SOME BASICS OF CLASS-SET THEORY 15
§1 Extensionality and separation 15
§2 Transitivity and supercompleteness 17
§3 Axiom of the empty set 18
§4 The pairing axiom 19
§5 The union axiom 21
§6 The power axiom 23
§7 Cartesian products 24
§8 Relations 24
§9 Functions 25
§10 Some useful facts about transitivity 26
§11 Basic universes 27

CHAPTER 3 THE NATURAL NUMBERS 29
§1 Preliminaries 29
§2 Definition of the natural numbers 32
§3 Derivation of the Peano postulates and other results 33
§4 A double induction principle and its applications 35
§5 Applications to natural numbers 40
§6 Finite sets 42
§7 Denumerable classes 42
§8 Definition by finite recursion 43

§9 Supplement—optional 44

CHAPTER 4 SUPERINDUCTION, WELL ORDERING AND CHOICE 47
§1 Introduction to well ordering 47
§2 Superinduction and double superinduction 52
§3 The well ordering of g-towers 56
§4 Well ordering and choice 57
§5 Maximal principles 61
§6 Another approach to maximal principles 64
§7 Cowen's theorem 66
§8 Another characterization of g-sets 68

CHAPTER 5 ORDINAL NUMBERS 71
§1 Ordinal numbers 71
§2 Ordinals and transitivity 75
§3 Some ordinals 76

CHAPTER 6 ORDER ISOMORPHISM AND TRANSFINITE RECURSION 79
§1 A few preliminaries 79
§2 Isomorphisms of well orderings 80
§3 The axiom of substitution 82
§4 The counting theorem 83
§5 Transfinite recursion theorems 84
§6 Ordinal arithmetic 88

CHAPTER 7 RANK 91
§1 The notion of rank 91
§2 Ordinal hierarchies 92
§3 Applications to the R_α sequence 93
§4 Zermelo universes 96

CHAPTER 8 FOUNDATION, $\in$-INDUCTION, AND RANK 99
§1 The notion of well-foundedness 99
§2 Descending $\in$-chains 100
§3 $\in$-Induction and rank 101
§4 Axiom E and Von Neumann's principle 103
§5 Some other characterizations of ordinals 104
§6 More on the axiom of substitution 106

CHAPTER 9 CARDINALS 107
§1 Some simple facts 107
§2 The Bernstein-Schröder theorem 108
§3 Denumerable sets 111
§4 Infinite sets and choice functions 111
§5 Hartog's theorem 112
§6 A fundamental theorem 114
§7 Preliminaries 116
§8 Cardinal arithmetic 118
§9 Sierpiński's theorem 121

II CONSISTENCY OF THE CONTINUUM HYPOTHESIS 125

CHAPTER 10 MOSTOWSKI-SHEPHERDSON MAPPINGS 127
§1 Relational systems 127
§2 Generalized induction and Γ-rank 129
§3 Generalized transfinite recursion 133
§4 Mostowski-Shepherdson maps 134
§5 More on Mostowski-Shepherdson mappings 136
§6 Isomorphisms, Mostowski-Shepherdson, well orderings 137

CHAPTER 11 REFLECTION PRINCIPLES 141
§0 Preliminaries 141
§1 The Tarski-Vaught theorem 145
§2 We add extensionality considerations 147
§3 The class version of the Tarski-Vaught theorems 148
§4 Mostowski, Shepherdson, Tarski, and Vaught 150
§5 The Montague-Levy reflection theorem 151

CHAPTER 12 CONSTRUCTIBLE SETS 155
§0 More on first-order definability 155
§1 The class L of constructible sets 156
§2 Absoluteness 158
§3 Constructible classes 163

CHAPTER 13 L IS A WELL FOUNDED FIRST-ORDER UNIVERSE 169
§1 First-order universes 169
§2 Some preliminary theorems about first-order universes 172
§3 More on first-order universes 174
§4 Another result 177

CHAPTER 14 CONSTRUCTIBILITY IS ABSOLUTE OVER L 179
§1 Σ-formulas and upward absoluteness 179
§2 More on Σ definability 181
§3 The relation $y = \mathcal{F}(x)$ 183
§4 Constructibility is absolute over L 189
§5 Further results 190
§6 A proof that L can be well ordered 191

CHAPTER 15 CONSTRUCTIBILITY AND THE CONTINUUM HYPOTHESIS 193
§0 What we will do 193
§1 The key result 194
§2 Gödel's isomorphism theorem (optional) 196
§3 Some consequences of Theorem G 198
§4 Metamathematical consequences of Theorem G 199
§5 Relative consistency of the axiom of choice 200
§6 Relative consistency of GCH and AC in class-set theory 201

III FORCING AND INDEPENDENCE RESULTS 205

CHAPTER 16 FORCING, THE VERY IDEA 207
§1 What is forcing? . . . 207
§2 What is modal logic? . . . 209
§3 What is $S4$ and why do we care? . . . 213
§4 A classical embedding . . . 215
§5 The basic idea . . . 221

CHAPTER 17 THE CONSTRUCTION OF $S4$ MODELS FOR ZF 223
§1 What are the models? . . . 223
§2 About equality . . . 229
§3 The well founded sets are present . . . 233
§4 Four more axioms . . . 235
§5 The definability of forcing . . . 239
§6 The substitution axiom schema . . . 242
§7 The axiom of choice . . . 244
§8 Where we stand now . . . 247

CHAPTER 18 THE AXIOM OF CONSTRUCTIBILITY IS INDEPENDENT 249
§1 Introduction . . . 249
§2 Ordinals are well behaved . . . 249
§3 Constructible sets are well behaved too . . . 251
§4 A real $S4$ model, at last . . . 252
§5 Cardinals are sometimes well behaved . . . 253
§6 The status of the generalized continuum hypothesis . . . 256

CHAPTER 19 INDEPENDENCE OF THE CONTINUUM HYPOTHESIS 259
§1 Power politics . . . 259
§2 The model . . . 259
§3 Cardinals stay cardinals . . . 260
§4 CH is independent . . . 262
§5 Cleaning it up . . . 263
§6 Wrapping it up . . . 266

CHAPTER 20 INDEPENDENCE OF THE AXIOM OF CHOICE 267
§1 A little history . . . 267
§2 Automorphism groups . . . 268
§3 Automorphisms preserve truth . . . 270
§4 Model and submodel . . . 272
§5 Verifying the axioms . . . 273
§6 AC is independent . . . 278

CHAPTER 21 CONSTRUCTING CLASSICAL MODELS 285
§1 On countable models . . . 285
§2 Cohen's way . . . 286
§3 Dense sets, filters, and generic sets . . . 287
§4 When generic sets exist . . . 289
§5 Generic extensions . . . 291
§6 The truth lemma . . . 294

§7 Conclusion 296

CHAPTER 22 FORCING BACKGROUND 299
§1 Introduction 299
§2 Cohen's version(s) 300
§3 Boolean valued models 301
§4 Unramified forcing 301
§5 Extensions 302

BIBLIOGRAPHY 305

INDEX 307

LIST OF NOTATION 313