

TEXTBOOKS in MATHEMATICS

Measure Theory and Fine Properties of Functions, Revised Edition provides a detailed examination of the central assertions of measure theory in n -dimensional Euclidean space. The book emphasizes the roles of Hausdorff measure and capacity in characterizing the fine properties of sets and functions.

Topics covered include a quick review of abstract measure theory, theorems and differentiation in \mathbb{R}^n , Hausdorff measures, area and coarea formulas for Lipschitz mappings and related change-of-variable formulas, and Sobolev functions as well as functions of bounded variation.

The text provides complete proofs of many key results omitted from other books, including Besicovitch's covering theorem, Rademacher's theorem (on the differentiability a.e. of Lipschitz functions), area and coarea formulas, the precise structure of Sobolev and BV functions, the precise structure of sets of finite perimeter, and Aleksandrov's theorem (on the twice differentiability a.e. of convex functions).

This revised edition includes countless improvements in notation, format, and clarity of exposition. Also new are several sections describing the π - λ theorem, weak compactness criteria in L^1 , and Young measure methods for weak convergence. In addition, the bibliography has been updated.

Topics are carefully selected and the proofs are succinct, but complete. This book provides ideal reading for mathematicians and for graduate students in pure and applied mathematics.



CRC Press
Taylor & Francis Group
an informa business

www.crcpress.com

K23386

ISBN 978-1-4822-4238-6



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