

# Contents

# Preface

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J		Jacobian matrix of transformation	
$J^{-1}$		inverse Jacobian matrix of transformation	
L		length of curve	
$n, n_i$		normal vector to surface	
P		point	
$P(n, k)$		partial derivative with respect to following index(es)	
$q^i$		generalized coordinate	
$q_i$		generalized velocity	
r		position vector	
R		Ricci curvature scalar	
$R_{ij}, R^i_j$		Ricci curvature tensor	
$R^i_{jkm}, R^i_{jkm}$		Riemann curvature tensor	
$r, \theta, \phi$		coordinates	
S		surface	
$S, S_{ij}$		rate of strain tensor	
$\bar{S}, \bar{S}_{ij}$		vorticity tensor	
t		time	
T (superscript)		Cauchy strain tensor	
$T, T_i$		traction vector	
tr		trace	
$w^i$		displacement vector	
$v, v_i$		velocity vector	
V		volume	
w		width	
$x_1, x^1$		Cartesian coordinates	
$x_2, x^2$		Cartesian coordinates	
$x, x^i$		vector of orthonormal Cartesian basis set	
$\gamma, \gamma_i$		metric tensor	
$\dot{\gamma}$		rate of strain tensor	
$\Gamma^k_{ij}$		Christoffel symbol of 2 <sup>nd</sup> kind	
$\delta$		Kronecker delta	
$\delta_{ij}, \delta^{ij}, \delta^i_j$		covariant, contravariant, and mixed Kronecker delta	
$\delta^i_j, \delta^j_i, \delta^k_l, \delta^l_k$		generalized Kronecker delta	
$\Delta, \Delta_i$		orthonormalized covariant basis vector	