

# CONTENTS

INTRODUCTION TO CHAPTERS IV, V AND VI	V
CONTENTS	VII
<hr/>	
<b>CHAPTER IV COXETER GROUPS AND TITS SYSTEMS</b>	
<hr/>	
§ 1. Coxeter Groups	1
1. Length and reduced decompositions	1
2. Dihedral groups	2
3. First properties of Coxeter groups	4
4. Reduced decompositions in a Coxeter group	5
5. The exchange condition	7
6. Characterisation of Coxeter groups	10
7. Families of partitions	10
8. Subgroups of Coxeter groups	12
9. Coxeter matrices and Coxeter graphs	13
§ 2. Tits Systems	15
1. Definitions and first properties	15
2. An example	17
3. Decomposition of $G$ into double cosets	18
4. Relations with Coxeter systems	19
5. Subgroups of $G$ containing $B$	21
6. Parabolic subgroups	22
7. The simplicity theorem	23
Appendix. Graphs	27
1. Definitions	27
2. The connected components of a graph	27
3. Forests and trees	29
Exercises for § 1.	31
Exercises for § 2.	44

---

**CHAPTER V GROUPS GENERATED BY REFLECTIONS**


---

<b>§ 1. Hyperplanes, chambers and facets</b> .....	61
1. Notations .....	61
2. Facets .....	62
3. Chambers .....	64
4. Walls and faces .....	65
5. Intersecting hyperplanes .....	67
6. Simplicial cones and simplices .....	68
<b>§ 2. Reflections</b> .....	70
1. Pseudo-reflections .....	70
2. Reflections .....	71
3. Orthogonal reflections .....	73
4. Orthogonal reflections in a euclidean affine space .....	73
5. Complements on plane rotations .....	74
<b>§ 3. Groups of displacements generated by reflections</b> .....	76
1. Preliminary results .....	77
2. Relation with Coxeter systems .....	78
3. Fundamental domain, stabilisers .....	79
4. Coxeter matrix and Coxeter graph of $W$ .....	81
5. Systems of vectors with negative scalar products .....	82
6. Finiteness theorems .....	84
7. Decomposition of the linear representation of $W$ on $T$ .....	86
8. Product decomposition of the affine space $E$ .....	88
9. The structure of chambers .....	89
10. Special points .....	91
<b>§ 4. The geometric representation of a Coxeter group</b> .....	94
1. The form associated to a Coxeter group .....	94
2. The plane $E_{s,s'}$ and the group generated by $\sigma_s$ and $\sigma_{s'}$ .....	95
3. The group and representation associated to a Coxeter matrix .....	96
4. The contragredient representation .....	97
5. Proof of lemma 1 .....	99
6. The fundamental domain of $W$ in the union of the chambers .....	101
7. Irreducibility of the geometric representation of a Coxeter group .....	102
8. Finiteness criterion .....	103
9. The case in which $B_M$ is positive and degenerate .....	105

---

**CHAPTER V GROUPS GENERATED BY REFLECTIONS**


---

<b>§ 1. Hyperplanes, chambers and facets</b> .....	61
1. Notations .....	61
2. Facets .....	62
3. Chambers .....	64
4. Walls and faces .....	65
5. Intersecting hyperplanes .....	67
6. Simplicial cones and simplices .....	68
<b>§ 2. Reflections</b> .....	70
1. Pseudo-reflections .....	70
2. Reflections .....	71
3. Orthogonal reflections .....	73
4. Orthogonal reflections in a euclidean affine space .....	73
5. Complements on plane rotations .....	74
<b>§ 3. Groups of displacements generated by reflections</b> .....	76
1. Preliminary results .....	77
2. Relation with Coxeter systems .....	78
3. Fundamental domain, stabilisers .....	79
4. Coxeter matrix and Coxeter graph of $W$ .....	81
5. Systems of vectors with negative scalar products .....	82
6. Finiteness theorems .....	84
7. Decomposition of the linear representation of $W$ on $T$ .....	86
8. Product decomposition of the affine space $E$ .....	88
9. The structure of chambers .....	89
10. Special points .....	91
<b>§ 4. The geometric representation of a Coxeter group</b> .....	94
1. The form associated to a Coxeter group .....	94
2. The plane $E_{s,s'}$ and the group generated by $\sigma_s$ and $\sigma_{s'}$ .....	95
3. The group and representation associated to a Coxeter matrix .....	96
4. The contragredient representation .....	97
5. Proof of lemma 1 .....	99
6. The fundamental domain of $W$ in the union of the chambers .....	101
7. Irreducibility of the geometric representation of a Coxeter group .....	102
8. Finiteness criterion .....	103
9. The case in which $B_M$ is positive and degenerate .....	105

<b>§ 3. Exponential invariants</b> .....	194
1. The group algebra of a free abelian group .....	194
2. Case of the group of weights: maximal terms .....	195
3. Anti-invariant elements .....	196
4. Invariant elements .....	199
<b>§ 4. Classification of root systems</b> .....	201
1. Finite Coxeter groups .....	201
2. Dynkin graphs .....	207
3. Affine Weyl group and completed Dynkin graph .....	210
4. Preliminaries to the construction of root systems .....	212
5. Systems of type $B_l$ ( $l \geq 2$ ) .....	214
6. Systems of type $C_l$ ( $l \geq 2$ ) .....	216
7. Systems of type $A_l$ ( $l \geq 1$ ) .....	217
8. Systems of type $D_l$ ( $l \geq 3$ ) .....	220
9. System of type $F_4$ .....	223
10. System of type $E_8$ .....	225
11. System of type $E_7$ .....	227
12. System of type $E_6$ .....	229
13. System of type $G_2$ .....	231
14. Irreducible non-reduced root systems .....	233
Exercises for § 1. ....	235
Exercises for § 2. ....	240
Exercises for § 3. ....	241
Exercises for § 4. ....	242
<b>HISTORICAL NOTE (Chapters IV, V and VI)</b> .....	249
<b>BIBLIOGRAPHY</b> .....	255
<b>INDEX OF NOTATION</b> .....	259
<b>INDEX OF TERMINOLOGY</b> .....	261
<b>PLATE I. Systems of type <math>A_l</math> (<math>l \geq 1</math>)</b> .....	265
<b>PLATE II. Systems of type <math>B_l</math> (<math>l \geq 2</math>)</b> .....	267
<b>PLATE III. Systems of type <math>C_l</math> (<math>l \geq 2</math>)</b> .....	269
<b>PLATE IV. Systems of type <math>D_l</math> (<math>l \geq 3</math>)</b> .....	271
<b>PLATE V. System of type <math>E_6</math></b> .....	275
<b>PLATE VI. System of type <math>E_7</math></b> .....	279
<b>PLATE VII. System of type <math>E_8</math></b> .....	283

PLATE VIII. System of type $F_4$ .....	287
PLATE IX. System of type $G_2$ .....	289
PLATE X. Irreducible systems of rank 2 .....	291
Summary of the principal properties of root systems .....	293

## 1. COXETER GROUPS

In this section,  $W$  denotes a group written multiplicatively, with identity element  $1$ , and  $S$  denotes a set of generators of  $W$  such that  $S = S^{-1}$  and  $1 \notin S$ . Every element of  $W$  is the product of a finite sequence of elements of  $S$ . From no. 3 onwards we assume that every element of  $S$  is of order 2.

### 1. LENGTH AND REDUCED DECOMPOSITIONS

**DEFINITION 1.** Let  $w \in W$ . The length of  $w$  (with respect to  $S$ ), denoted by  $l_S(w)$  or simply by  $l(w)$ , is the smallest integer  $q \geq 0$  such that  $w$  is the product of a sequence of  $q$  elements of  $S$ . A reduced decomposition of  $w$  (with respect to  $S$ ) is any sequence  $s = (s_1, \dots, s_q)$  of elements of  $S$  such that  $w = s_1 \dots s_q$  and  $q = l(w)$ .

Thus  $1$  is the unique element of length 0 and  $S$  consists of the elements of length 1.

**PROPOSITION 1.** Let  $w$  and  $w'$  be in  $W$ . We have the formulas:

$$l(ww') \leq l(w) + l(w'), \quad (1)$$

$$l(w^{-1}) = l(w), \quad (2)$$

$$|l(w) - l(w')| \leq l(ww'^{-1}). \quad (3)$$

Let  $(s_1, \dots, s_p)$  and  $(s'_1, \dots, s'_q)$  be reduced decompositions of  $w$  and  $w'$  respectively. Thus  $l(w) = p$  and  $l(w') = q$ . Since  $ww' = s_1 \dots s_p s'_1 \dots s'_q$ , we have  $l(ww') \leq p + q$ , proving (1). Since  $S = S^{-1}$  and  $w^{-1} = s_p^{-1} \dots s_1^{-1}$ , we have  $l(w^{-1}) \leq p = l(w)$ . Replacing  $w$  by  $w^{-1}$  gives the opposite inequality  $l(w) \leq l(w^{-1})$ , proving (2). Replacing  $w$  by  $ww'^{-1}$  in (1) and (2) gives the relations

$$l(w) + l(w') \leq l(ww'^{-1}), \quad (4)$$

$$l(ww'^{-1}) = l(w'w^{-1}). \quad (5)$$