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Consider an n -dimensional vector space V over the field \mathbb{K} of real or complex numbers. Its dual space V^* consists of all linear maps from V to \mathbb{K} . More generally, a multilinear and antisymmetric map

$$\omega^k : V \times \dots \times V \rightarrow \mathbb{K},$$

depending on k vectors from the vector space V , is called an exterior (multilinear) form of degree k . The antisymmetry of ω^k means that, for all k vectors v_1, \dots, v_k from V and any permutation $\sigma \in S_k$ of the numbers $\{1, \dots, k\}$, the following equation holds:

$$\omega^k(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma) \cdot \omega^k(v_1, \dots, v_k).$$

Here $\text{sgn}(\sigma)$ denotes the sign of the permutation σ . In particular, ω^k changes sign under a transposition of the indices i and j :

$$\omega^k(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega^k(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

The vector space of all exterior k -forms will be denoted by $\Lambda^k(V^*)$. Furthermore, we will use the conventions $\Lambda^0(V^*) = \mathbb{K}$ and $\Lambda^1(V^*) = V^*$.

Fixing an arbitrary basis e_1, \dots, e_n in the n -dimensional vector space V , we see that each exterior k -form ω^k is uniquely determined by its values on all k -tuples of the form e_{i_1}, \dots, e_{i_k} , where the indices are always supposed to be strictly ordered, $I = (i_1 < \dots < i_k)$. On the other hand, a k -form can be defined by arbitrarily prescribing its values on all ordered k -tuples of basis vectors and extending it to all k -tuples of vectors in an antisymmetric and