

# Contents

## Chapter 1

### Introduction

1.1.	Introductory remarks . . . . .	1
1.2.	The plan of the book: notation . . . . .	2
1.3.	Very brief historical remarks . . . . .	5
1.4.	The EULER equations . . . . .	7
1.5.	Other classical necessary conditions . . . . .	10
1.6.	Classical sufficient conditions . . . . .	12
1.7.	The direct methods . . . . .	15
1.8.	Lower semicontinuity . . . . .	19
1.9.	Existence. . . . .	23
1.10.	The differentiability theory. Introduction . . . . .	26
1.11.	Differentiability; reduction to linear equations . . . . .	34

## Chapter 2

### Semi-classical results

2.1.	Introduction . . . . .	39
2.2.	Elementary properties of harmonic functions . . . . .	40
2.3.	WEYL's lemma . . . . .	41
2.4.	POISSON's integral formula; elementary functions; GREEN's functions . . . . .	43
2.5.	Potentials . . . . .	47
2.6.	Generalized potential theory; singular integrals . . . . .	48
2.7.	The CALDERON-ZYGMUND inequalities . . . . .	55
2.8.	The maximum principle for a linear elliptic equation of the second order . . . . .	61

## Chapter 3

### The spaces $H_p^m$ and $H_{p0}^m$

3.1.	Definitions and first theorems . . . . .	62
3.2.	General boundary values; the spaces $H_{p0}^m(G)$ ; weak convergence. . . . .	68
3.3.	The DIRICHLET problem . . . . .	70
3.4.	Boundary values . . . . .	72
3.5.	Examples; continuity; some SOBOLEV lemmas. . . . .	78
3.6.	Miscellaneous additional results . . . . .	81
3.7.	Potentials and quasi-potentials; generalizations . . . . .	86

## Chapter 4

## Existence theorems

- |      |   |     |
|------|---|-----|
| 4.1. | The lower-semicontinuity theorems of SERRIN . . . . .   | 90  |
| 4.2. | Variational problems with $f = f(p)$ ; the equations (1.10.13) with $N = 1$ ,<br>$B_i = 0$ , $A^\alpha = A^\alpha(p)$ . . . . . | 98  |
| 4.3. | The borderline cases $k = \nu$ . . . . .  | 105 |
| 4.4. | The general quasi-regular integral . . . . .  | 112 |

## Chapter 5

## Differentiability of weak solutions

- |       |  |     |
|-------|--|-----|
| 5.1.  | Introduction . . . . .   | 126 |
| 5.2.  | General theory; $\nu > 2$ . . . . .  | 128 |
| 5.3.  | Extensions of the DE GIORGI-NASH-MOSER results; $\nu > 2$ . . . . .                  | 134 |
| 5.4.  | The case $\nu = 2$ . . . . .   | 143 |
| 5.5.  | $L_p$ and SCHAUDER estimates . . . . .   | 149 |
| 5.6.  | The equation $a \cdot \nabla^2 u + b \cdot \nabla u + c u - \lambda u = f$ . . . . . | 157 |
| 5.7.  | Analyticity of the solutions of analytic linear equations . . . . .                  | 164 |
| 5.8.  | Analyticity of the solutions of analytic, non-linear, elliptic equations . . . . .   | 170 |
| 5.9.  | Properties of the extremals; regular cases . . . . .                                 | 186 |
| 5.10. | The extremals in the case $1 < k < 2$ . . . . .                                      | 191 |
| 5.11. | The theory of LADYZENSKAYA and URAL'TSEVA . . . . .                                  | 194 |
| 5.12. | A class of non-linear equations . . . . .  | 203 |

## Chapter 6

## Regularity theorems for the solutions of general elliptic systems and boundary value problems

- |      |   |     |
|------|---|-----|
| 6.1. | Introduction . . . . .  | 209 |
| 6.2. | Interior estimates for general elliptic systems . . . . .   | 215 |
| 6.3. | Estimates near the boundary; coerciveness . . . . .   | 225 |
| 6.4. | Weak solutions . . . . .  | 242 |
| 6.5. | The existence theory for the DIRICHLET problem for strongly elliptic systems . . . . .                                  | 251 |
| 6.6. | The analyticity of the solutions of analytic systems of linear elliptic equations . . . . .                             | 258 |
| 6.7. | The analyticity of the solutions of analytic nonlinear elliptic systems . . . . .                                       | 266 |
| 6.8. | The differentiability of the solutions of non-linear elliptic systems; weak solutions; a perturbation theorem . . . . . | 277 |

## Chapter 7

## A variational method in the theory of harmonic integrals

- |      |   |     |
|------|---|-----|
| 7.1. | Introduction . . . . .  | 286 |
| 7.2. | Fundamentals; the GAFFNEY-GÄRDING inequality . . . . .                                    | 288 |
| 7.3. | The variational method . . . . .  | 293 |
| 7.4. | The decomposition theorem. Final results for compact manifolds without boundary . . . . . | 295 |
| 7.5. | Manifolds with boundary . . . . .   | 300 |
| 7.6. | Differentiability at the boundary . . . . .   | 305 |
| 7.7. | Potentials, the decomposition theorem . . . . .   | 309 |
| 7.8. | Boundary value problems . . . . .   | 314 |

## Chapter 8

The  $\bar{\partial}$ -NEUMANN problem on strongly pseudo-convex manifolds

8.1. Introduction . . . . .	316
8.2. Results. Examples. The analytic embedding theorem . . . . .	320
8.3. Some important formulas . . . . .	328
8.4. The HILBERT space results . . . . .	333
8.5. The local analysis . . . . .	337
8.6. The smoothness results . . . . .	341

## Chapter 9

## Introduction to parametric Integrals; two dimensional problems

9.1. Introduction. Parametric integrals . . . . .	349
9.2. A lower semi-continuity theorem . . . . .	354
9.3. Two dimensional problems; introduction; the conformal mapping of surfaces . . . . .	362
9.4. The problem of PLATEAU . . . . .	374
9.5. The general two-dimensional parametric problem . . . . .	390

## Chapter 10

## The higher dimensional PLATEAU problems

10.1. Introduction . . . . .	400
10.2. $\nu$ surfaces, their boundaries, and their HAUSDORFF measures . . . . .	407
10.3. The topological results of ADAMS . . . . .	414
10.4. The minimizing sequence; the minimizing set . . . . .	421
10.5. The local topological disc property . . . . .	439
10.6. The REIFENBERG cone inequality . . . . .	459
10.7. The local differentiability . . . . .	474
10.8. Additional results of FEDERER concerning LEBESGUE $\nu$ -area . . . . .	480
Bibliography . . . . .	494
Index . . . . .	504