Contents

Preface **xiii**

Author **xvii**

I Solving Linear and Nonlinear Systems of Equations **1**

1 Solving Linear Systems Using Direct Methods **3**
- 1.1 Testing the Existence of the Solution 3
- 1.2 Methods for Solving Linear Systems 5
  - 1.2.1 Special Linear Systems 5
  - 1.2.2 Gauss and Gauss-Jordan Elimination 9
  - 1.2.3 Solving the System with the rref Function 10
- 1.3 Matrix Factorization Techniques 12
  - 1.3.1 The LU Factorization 12
  - 1.3.2 The QR Factorization 16
  - 1.3.3 The Singular Value Decomposition (SVD) 19

2 Solving Linear Systems with Iterative and Least Squares Methods **23**
- 2.1 Mathematical Backgrounds 23
  - 2.1.1 Convergent Sequences and Cauchi's Convergence . . . 23
  - 2.1.2 Vector Norm 25
  - 2.1.3 Convergent Sequences of Vectors 25
- 2.2 The Iterative Methods 25
  - 2.2.1 The General Idea 26
  - 2.2.2 The Jacobi Method 28
  - 2.2.3 The Jacobi Method in the Matrix Form 31
    - 2.2.3.1 The Gauss-Seidel Iterative Method 33
  - 2.2.4 The Gauss-Seidel Method in the Vector Form 35
  - 2.2.5 The Relaxation Methods 36
- 2.3 The Least Squares Solutions 39
  - 2.3.1 Some Applications of Least Squares Solutions . . . 43

3 **Ill-Conditioning and Regularization Techniques in Solutions of Linear Systems** **57**
3.1 Ill-Conditioning in Solutions of Linear Systems 57
3.1.1 More Examples of Ill-Posed System 63
3.1.2 Condition Numbers and Ill-Conditioned Matrices . . . 67
3.1.3 Linking the Condition Numbers to Matrix Related Eigenvalues 71
3.1.4 Further Analysis on Ill-Posed Systems 75
3.2 Regularization of Solutions in Linear Systems 78
3.2.1 The Truncated SVD (TSVD) Method 78
3.2.2 Tikhonov Regularizaton Method 82
3.2.3 The L-curve Method 87
3.2.4 The Discrepancy Principle 87

4 **Solving a System of Nonlinear Equations** **89**
4.1 Solving a Single Nonlinear Equation 89
4.1.1 The Bisection Method 89
4.1.2 The Newton-Raphson Method 91
4.1.3 The Secant Method 93
4.1.4 The Iterative Method Towards a Fixed Point 94
4.1.5 Using the MATLAB and Python `solve` Function . . . 96
4.2 Solving a System of Nonlinear Equations 97

II **Data Interpolation and Solutions of Differential Equations** **103**

5 **Data Interpolation** **105**
5.1 Lagrange Interpolation 105
5.1.1 Construction of Lagrange Interpolating Polynomial . . 105
5.1.2 Uniqueness of Lagrange Interplation Polynomial . . . 107
5.1.3 Lagrange Interpolation Error 108
5.2 Newton's Interpolation 109
5.2.1 Description of the Method 109
5.2.2 Newton's Divided Differences 113
5.3 MATLAB's Interpolation Tools 116
5.3.1 Interpolation with the `interp1` Function 116
5.3.2 Interpolation with the Spline Function 117
5.3.3 Interpolation with the Function pchip 119
5.3.4 Calling the Functions `spline` and `pchip` from `interp1` 120
5.4 Data Interpolation in Python 120
5.4.1 The Function `interp1d` 121
5.4.2 The Functions `pchip_interpolate` and `CubicSpline` 122
5.4.3 The Function `lagrange` 123

6 Numerical Differentiation and Integration 125
6.1 Numerical Differentiation 125
6.1.1 Approximating Derivatives with Finite Differences . . 125
6.2 Numerical Integration 133
6.2.1 Newton-Cotes Methods 134
6.2.2 The Gauss Integration Method 143

7 Solving Systems of Nonlinear Ordinary Differential Equations 165
7.1 Runge-Kutta Methods 165
7.2 Explicit Runge-Kutta Methods 167
7.2.1 Euler's Method 168
7.2.2 Heun's Method 171
7.2.3 The Fourth-Order Runge-Kutta Method 174
7.3 Implicit Runge-Kutta Methods 176
7.3.1 The Backward Euler Method 176
7.3.2 Collocation Runge-Kutta Methods 180
7.3.2.1 Legendre-Gauss Methods 180
7.3.2.2 Lobatto Methods 184
7.4 MATLAB ODE Solvers 191
7.4.1 MATLAB ODE Solvers 191
7.4.2 Solving a Single IVP 191
7.4.3 Solving a System of IVPs 193
7.4.4 Solving Stiff Systems of IVPs 195
7.5 Python Solvers for IVPs 197
7.5.1 Solving ODEs with odeint 197
7.5.2 Solving ODEs with Gekko 201

8 Nonstandard Finite Difference Methods for Solving ODEs 207
8.1 Deficiencies with Standard Finite Difference Schemes 207
8.2 Construction Rules of Nonstandard Finite Difference Schemes 213
8.3 Exact Finite Difference Schemes 217
8.3.1 Exact Finite Difference Schemes for Homogeneous Linear ODEs 218
8.3.1.1 Exact Finite Difference Schemes for a Linear Homogeneous First-Order ODE 218
8.3.1.2 Exact Finite Difference Scheme for Linear Homogeneous Second Order ODE 220
8.3.1.3 Exact Finite Difference Scheme for a System of Two Linear ODEs 223
8.3.2 Exact Difference Schemes for Nonlinear Equations . . 230
8.3.3 Exact Finite Difference Schemes for Differential Equations with Linear and Power Terms 234
8.4 Other Nonstandard Finite Difference Schemes 236

III Solving Linear, Nonlinear and Dynamic Optimization Problems 241

9 Solving Optimization Problems: Linear and Quadratic Programming 243

9.1 Form of a Linear Programming Problem 243

9.2 Solving Linear Programming Problems with `linprog` 246

9.3 Solving Linear Programming Problems with `fmincon` MATLAB's Functions 249

9.4 Solving Linear Programming Problems with `pulp` Python . . 250

9.5 Solving Linear Programming Problems with `pyomo` . . . 252

9.6 Solving Linear Programming Problems with `gekko` 254

9.7 Solving Quadratic Programming Problems 255

10 Solving Optimization Problems: Nonlinear Programming 261

10.1 Solving Unconstrained Problems 261

10.1.1 Line Search Algorithm 263

10.1.2 The Steepest Descent Method 264

10.1.3 Newton's Method 266

10.1.4 Quasi Newton's Methods 269

10.1.4.1 The Broyden-Fletcher-Goldfarb-Shanno (BFGS) Method 269

10.1.4.2 The Davidon-Fletcher-Powell (DFP) Algorithm 272

10.1.5 Solving Unconstrained Optimization Problems with MATLAB 274

10.1.6 Solving an Unconstrained Problem with Python . . 275

10.1.7 Solving Unconstrained Optimization Problems with Gekko 276

10.2 Solving Constrained Optimization Problems 278

10.2.1 Solving Constrained Optimization Problems with MATLAB `fmincon` Function 280

10.2.2 Solving Constrained Minimization Problems in Python 284

10.2.3 Solving Constrained Optimization with Gekko Python 286

11 Solving Optimal Control Problems 289

11.1 Introduction 289

11.2 The First-Order Optimality Conditions and Existence of Optimal Control 290

11.3 Necessary Conditions of the Discretized System 293

11.4 Numerical Solution of Optimal Control 294

11.5 Solving Optimal Control Problems Using Indirect Methods . 295

11.5.1 Numerical Solution Using Indirect Transcription Method 295

11.6 Solving Optimal Control Problems Using Direct Methods . . 306
11.6.1 Statement of the Problem 307
11.6.2 The Control Parameterization Technique 307
11.6.2.1 Examples 309
11.6.3 The Gekko Python Solver 313

Bibliography **321**

Index **327**