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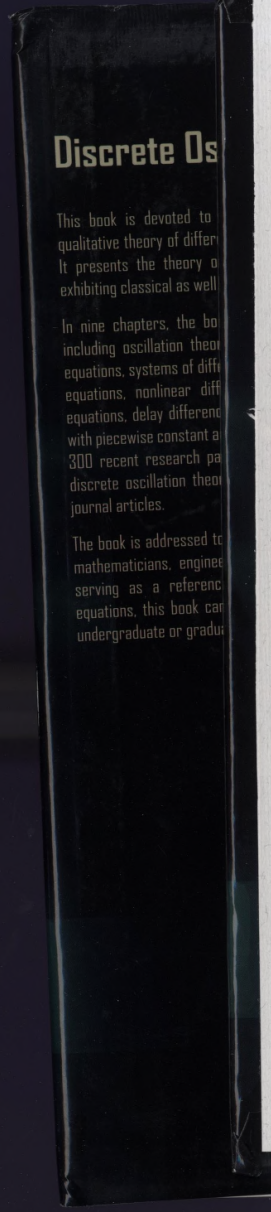

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In nine chapters, the boo
including oscillation theor
equations, systems of diffe
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with piecewise constant arg
300 recent research pap
discrete oscillation theory
journal articles.

The book is addressed to a
mathematicians, engineers
serving as a reference
equations, this book can a
undergraduate or graduate