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It is devoted to the Fibonacci numbers. We start with the familiar definition, move on to some more sophisticated points of view, and then formulate some questions that are typical of those that can be addressed using the material of this book.

1.1 The Rabbit Problem

In the year 1202 the Italian mathematician Leonardo Pisano (which means Leonardo of Pisa) published *Liber Abaci*,¹ a book of problems whose purpose was to illustrate the usefulness of Arabic numerals in arithmetic computations because at that time cumbersome Roman numerals were still being used in Italy. One of the problems discussed in Pisano's book considers pairs of breeding rabbits. Each pair of rabbits matures in two months and produces one new pair each month thereafter, beginning with the last day of its second month. Starting with a single infant pair born at the beginning of Month 0, how many pairs will there be one year after this pair begins breeding? We can find our way to a solution by considering what happens in the first few months:

At the end of Month 0 there is only one pair, and they are not yet breeding.

¹The book was reprinted in 1857–1862 by Baldassarre Boncompagni [14]. A translation by L. Sigler [147] has recently been published by Springer-Verlag.