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If we allow for some randomness in some of the coefficients of a differential equation we often obtain a more realistic mathematical model of the situation.

Problem 1. Consider the simple population growth model

$$\frac{dN}{dt} = a(t)N(t), \quad N(0) = N_0 \text{ (constant)} \quad (1.1.1)$$

where $N(t)$ is the size of the population at time t , and $a(t)$ is the relative rate of growth at time t . It might happen that $a(t)$ is not completely known, but subject to some random environmental effects, so that we have

$$a(t) = r(t) + \text{"noise"}$$

where we do not know the exact behaviour of the noise term, only its probability distribution. The function $r(t)$ is assumed to be nonrandom. How do we solve (1.1.1) in this case?

Problem 2. The charge $Q(t)$ at time t at a fixed point in an electric circuit satisfies the differential equation

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = F(t), \quad Q(0) = Q_0, \quad Q'(0) = \dot{Q}_0 \quad (1.1.2)$$

where L is inductance, R is resistance, C is capacitance and $F(t)$ the potential source at time t .