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Some of the bibliographic references, for example in Chapters 4 and 6, are to Tibshirani, R.J., rather than Tibshirani, R.; the former is Ryan Tibshirani, the latter is Robert (son and father).