Contents

Preface ix

Part One 3

1 Category: The Essence of Composition 3
- 1.1 Arrows as Functions . . . 3
- 1.2 Properties of Composition . . . 5
- 1.3 Composition is the Essence of Programming . . . 7
- 1.4 Challenges . . . 8

2 Types and Functions 9
- 2.1 Who Needs Types? . . . 9
- 2.2 Types Are About Composability . . . 10
- 2.3 What Are Types? . . . 11
- 2.4 Why Do We Need a Mathematical Model? . . . 12
- 2.5 Pure and Dirty Functions . . . 14
- 2.6 Examples of Types . . . 15
- 2.7 Challenges . . . 17

3 Categories Great and Small 19
- 3.1 No Objects . . . 19
- 3.2 Simple Graphs . . . 19
- 3.3 Orders . . . 20
- 3.4 Monoid as Set . . . 20
- 3.5 Monoid as Category . . . 23
- 3.6 Challenges . . . 25

4 Kleisli Categories 27
- 4.1 The Writer Category . . . 30
- 4.2 Writer in Haskell . . . 32
- 4.3 Kleisli Categories . . . 34
- 4.4 Challenge . . . 34

5 Products and Coproducts 37
5.1 Initial Object . . . 37
5.2 Terminal Object . . . 39
5.3 Duality . . . 40
5.4 Isomorphisms . . . 40
5.5 Products . . . 42
5.6 Coproduct . . . 45
5.7 Asymmetry . . . 48
5.8 Challenges . . . 49
5.9 Bibliography . . . 50

6 Simple Algebraic Data Types 51
6.1 Product Types . . . 51
6.2 Records . . . 54
6.3 Sum Types . . . 55
6.4 Algebra of Types . . . 58
6.5 Challenges . . . 61

7 Functors 63
7.1 Functors in Programming . . . 65
7.1.1 The Maybe Functor . . . 65
7.1.2 Equational Reasoning . . . 66
7.1.3 Optional . . . 68
7.1.4 Typeclasses . . . 69
7.1.5 Functor in C++ . . . 70
7.1.6 The List Functor . . . 71
7.1.7 The Reader Functor . . . 72
7.2 Functors as Containers . . . 74
7.3 Functor Composition . . . 75
7.4 Challenges . . . 77

8 Functoriality 79
8.1 Bifunctors . . . 79
8.2 Product and Coproduct Bifunctors . . . 81
8.3 Functorial Algebraic Data Types . . . 82
8.4 Functors in C++ . . . 85
8.5 The Writer Functor . . . 86
8.6 Covariant and Contravariant Functors . . . 87
8.7 Profunctors . . . 89
8.8 The Hom-Functor . . . 90
8.9 Challenges . . . 91

9 Function Types 93
9.1 Universal Construction . . . 94
9.2 Currying . . . 97
9.3 Exponentials . . . 100
9.4 Cartesian Closed Categories . . . 101
9.5 Exponentials and Algebraic Data Types . . . 101
9.5.1 Zeroth Power . . . 101
9.5.2 Powers of One . . . 102
9.5.3 First Power . . . 102
9.5.4 Exponentials of Sums . . . 102
9.5.5 Exponentials of Exponentials . . . 103
9.5.6 Exponentials over Products . . . 103
9.6 Curry-Howard Isomorphism . . . 103
9.7 Bibliography . . . 105

10 Natural Transformations 107
10.1 Polymorphic Functions . . . 110
10.2 Beyond Naturality . . . 114
10.3 Functor Category . . . 115
10.4 2-Categories . . . 118
10.5 Conclusion . . . 121
10.6 Challenges . . . 121

Part Two 125

11 Declarative Programming 125

12 Limits and Colimits 131
12.1 Limit as a Natural Isomorphism . . . 135
12.2 Examples of Limits . . . 138
12.3 Colimits . . . 142
12.4 Continuity . . . 143
12.5 Challenges . . . 145

13 Free Monoids 147
13.1 Free Monoid in Haskell . . . 148
13.2 Free Monoid Universal Construction . . . 149
13.3 Challenges . . . 152

14 Representable Functors 153
14.1 The Hom Functor . . . 154
14.2 Representable Functors . . . 156
14.3 Challenges . . . 159
14.4 Bibliography . . . 159

15 The Yoneda Lemma **161**
15.1 Yoneda in Haskell 165
15.2 Co-Yoneda 167
15.3 Challenges 167
15.4 Bibliography 168

16 Yoneda Embedding **169**
16.1 The Embedding 170
16.2 Application to Haskell 171
16.3 Preorder Example 172
16.4 Naturality 173
16.5 Challenges 174

Part Three **177**

17 It's All About Morphisms **177**
17.1 Functors 177
17.2 Commuting Diagrams 177
17.3 Natural Transformations 178
17.4 Natural Isomorphisms 179
17.5 Hom-Sets 180
17.6 Hom-Set Isomorphisms 180
17.7 Asymmetry of Hom-Sets 181
17.8 Challenges 181

18 Adjunctions **183**
18.1 Adjunction and Unit/Counit Pair 183
18.2 Adjunctions and Hom-Sets 187
18.3 Product from Adjunction 190
18.4 Exponential from Adjunction 193
18.5 Challenges 194

19 Free/Forgetful Adjunctions **195**
19.1 Some Intuitions 197
19.2 Challenges 199

20 Monads: Programmer's Definition **201**
20.1 The Kleisli Category 202
20.2 Fish Anatomy 204
20.3 The do Notation 205

21 Monads and Effects **209**
21.1 The Problem . . . 209
21.2 The Solution . . . 210
21.2.1 Partiality . . . 210
21.2.2 Nondeterminism . . . 211
21.2.3 Read-Only State . . . 212
21.2.4 Write-Only State . . . 214
21.2.5 State . . . 214
21.2.6 Exceptions . . . 215
21.2.7 Continuations . . . 215
21.2.8 Interactive Input . . . 217
21.2.9 Interactive Output . . . 218
21.3 Conclusion . . . 219

22 Monads Categorically **221**
22.1 Monoidal Categories . . . 224
22.2 Monoid in a Monoidal Category . . . 228
22.3 Monads as Monoids . . . 229
22.4 Monads from Adjunctions . . . 230

23 Comonads **233**
23.1 Programming with Comonads . . . 234
23.2 The Product Comonad . . . 234
23.3 Dissecting the Composition . . . 235
23.4 The Stream Comonad . . . 236
23.5 Comonad Categorically . . . 238
23.6 The Store Comonad . . . 240
23.7 Challenges . . . 241

24 F-Algebras **243**
24.1 Recursion . . . 245
24.2 Category of F-Algebras . . . 247
24.3 Natural Numbers . . . 249
24.4 Catamorphisms . . . 250
24.5 Folds . . . 251
24.6 Coalgebras . . . 252
24.7 Challenges . . . 254

25 Algebras for Monads **257**
25.1 T-algebras . . . 259
25.2 The Kleisli Category . . . 261
25.3 Coalgebras for Comonads . . . 263
25.4 Lenses . . . 263
25.5 Challenges . . . 265

26 Ends and Coends **267**
26.1 Dinatural Transformations . . . 268
26.2 Ends . . . 269
26.3 Ends as Equalizers . . . 272
26.4 Natural Transformations as Ends . . . 272
26.5 Coends . . . 273
26.6 Ninja Yoneda Lemma . . . 276
26.7 Profunctor Composition . . . 277

27 Kan Extensions **279**
27.1 Right Kan Extension . . . 281
27.2 Kan Extension as Adjunction . . . 282
27.3 Left Kan Extension . . . 284
27.4 Kan Extensions as Ends . . . 286
27.5 Kan Extensions in Haskell . . . 288
27.6 Free Functor . . . 290

28 Enriched Categories **293**
28.1 Why Monoidal Category? . . . 294
28.2 Monoidal Category . . . 294
28.3 Enriched Category . . . 296
28.4 Preorders . . . 297
28.5 Metric Spaces . . . 298
28.6 Enriched Functors . . . 299
28.7 Self Enrichment . . . 300
28.8 Relation to 2-Categories . . . 301

29 Topoi **303**
29.1 Subobject Classifier . . . 304
29.2 Topos . . . 307
29.3 Topoi and Logic . . . 307
29.4 Challenges . . . 308

30 Lawvere Theories **309**
30.1 Universal Algebra . . . 309
30.2 Lawvere Theories . . . 310
30.3 Models of Lawvere Theories . . . 313
30.4 The Theory of Monoids . . . 314
30.5 Lawvere Theories and Monads . . . 315
30.6 Monads as Coends . . . 317
30.7 Lawvere Theory of Side Effects . . . 319
30.8 Challenges . . . 320
30.9 Further Reading . . . 320

31 Monads, Monoids, and Categories 323
31.1 Bicategories 323
31.2 Monads 327
31.3 Challenges 330
31.4 Bibliography 330

Appendices 331

Index 331

Acknowledgments 333

Colophon 334

Copyleft notice 335