Contents

Preface *page* xi

Introduction 1

0.1 The leitmotiv 1

0.2 A few words about the history of entropy 3

0.3 Multiple meanings of entropy 4

0.4 Conventions 18

PART I Entropy in ergodic theory 21

1 Shannon information and entropy 23

1.1 Information and entropy of probability vectors 23

1.2 Partitions and sigma-algebras 30

1.3 Information and static entropy of a partition 32

1.4 Conditional static entropy 33

1.5 Conditional entropy via probabilistic tools* 35

1.6 Basic properties of static entropy 36

1.7 Metrics on the space of partitions 42

1.8 Mutual information* 46

1.9 Non-Shannon inequalities* 48

Exercises 51

2 Dynamical entropy of a process 53

2.1 Subadditivity 53

2.2 Preliminaries on dynamical systems 57

2.3 Dynamical entropy of a process 60

2.4 Properties of dynamical entropy 65

2.5 Affinity of dynamical entropy 68

2.6 Conditional dynamical entropy via disintegration* 69

2.7 Summary of the properties of entropy 72
2.8 Combinatorial entropy 73
Exercises 78

3 Entropy theorems in processes 80
3.1 Independence and ε-independence 80
3.2 The Pinsker sigma-algebra in a process 85
3.3 The Shannon–McMillan–Breiman Theorem 89
3.4 The Ornstein–Weiss Return Times Theorem 94
3.5 Horizontal data compression 97
Exercises 100

4 Kolmogorov–Sinai Entropy 102
4.1 Entropy of a dynamical system 102
4.2 Generators 105
4.3 The natural extension 111
4.4 Joinings 116
4.5 Ornstein Theory* 120
Exercises 130

5 The Ergodic Law of Series* 132
5.1 History of the Law of Series 132
5.2 Attracting and repelling in signal processes 135
5.3 Decay of repelling in positive entropy 139
5.4 Typicality of attracting for long cylinders 152

PART II Entropy in topological dynamics 157

6 Topological entropy 159
6.1 Three definitions of topological entropy 159
6.2 Properties of topological entropy 165
6.3 Topological conditional and tail entropies 167
6.4 Properties of topological conditional entropy 171
6.5 Topological joinings 172
6.6 The simplex of invariant measures 175
6.7 Topological fiber entropy 179
6.8 The major Variational Principles 181
6.9 Determinism in topological systems 190
6.10 Topological preimage entropy* 197
Exercises 199

7 Dynamics in dimension zero 201
7.1 Zero-dimensional dynamical systems 201
7.2 Topological entropy in dimension zero 202
7.3 The invariant measures in dimension zero 203
7.4 The Variational Principle in dimension zero 205
7.5 Tail entropy and asymptotic h-expansiveness in dimension zero 206
7.6 Principal zero-dimensional extensions 212
Exercises 225

8 The entropy structure 227
8.1 The type of convergence 227
8.2 U.s.d.a.-sequences on simplices 244
8.3 Entropy of a measure with respect to a topological resolution 254
8.4 Entropy structure 263
Exercises 270

9 Symbolic extensions 272
9.1 What are symbolic extensions? 272
9.2 The Symbolic Extension Entropy Theorem 274
9.3 Properties of symbolic extension entropy 287
9.4 Symbolic extensions of interval maps 293
Exercises 301

10 A touch of smooth dynamics* 303
10.1 Margulis–Ruelle Inequality and Pesin Entropy Formula 303
10.2 Tail entropy estimate 307
10.3 Symbolic extensions of smooth systems 308

PART III Entropy theory for operators 311

11 Measure-theoretic entropy of stochastic operators 313
11.1 A few words on operator dynamics 313
11.2 The axiomatic measure-theoretic definition 316
11.3 An explicit measure-theoretic definition 329
11.4 Not so bad properties of the operator entropy 332
Exercises 335

12 Topological entropy of a Markov operator 336
12.1 Three definitions 336
12.2 Properties of the topological operator entropy 339

12.3 Half of the variational principle 341
Exercises 343

13 Open problems in operator entropy 344
13.1 Questions on doubly stochastic operators 344
13.2 Questions concerning Markov operators 345

***Appendix A* Toolbox** 347

***Appendix B* Conditional S–M–B** 366

List of symbols 374
References 379
Index 386