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We use a hierarchical numbering system for equations and statements. The  $i$ -th equation in Section  $j$  of Chapter  $k$  is labeled  $(j/k)$  at the place where it occurs, and is cited as  $(j/k)$  within Chapter  $k$ , but as  $(j/k)$  outside Chapter  $k$ . A definition, theorem, lemma, corollary, remark, problem, exercise, or solution is labeled “Statement  $k$ ,” and the  $k$ -th statement in Section  $j$  of Chapter  $i$  is labeled  $j/k$  Statement at the place where it occurs, and is cited as Statement  $j/k$  within Chapter  $i$  but as Statement  $j/k$  outside Chapter  $i$ .

This book is intended as a text and can be used in either a one-semester or a two-semester course, or as a text for a special topic session. The accompanying figure shows dependencies among sections, and in some cases among subsections. In a one-semester course, we recommend including all of Chapter 1 and Sections 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, §3.1 A, B, E, Sections 3.3, 3.4, 3.5, and §3.6 A, C. This material provides the basic theory of stochastic differential equations, including the Itô calculus and the basic existence and uniqueness results for strong solutions of stochastic differential equations. It also includes solutions of interest in engineering applications, namely, the Ornstein–Uhlenbeck integrals and approximation of stochastic differential equations in §3.4 and 3.2 D, and Gauss–Markov processes in §3.6 A. Progress through this material can be accelerated by omitting the proof of the Doob–Meyer Decomposition Theorem 1.4.10 and the proofs in §3.4 D. The remainder of Theorem 1.4.10, Theorem 1.4.20, Definition 2.4.11, and Remark 2.4.12 should, however, be retained. If possible in a one-semester course, and certainly in a two-semester course, one should include the topic of weak solutions of stochastic differential equations. This is accomplished by covering §4.1 A, B, and Sections 3.5, 3.3, and 3.4. Section 5.2 serves as an introduction to nonlinear control, and so we recommend adding §3.4 C, D, E, and Section 5.1, and §4.1 B if time permits. In either a one- or two-semester course, Section 1.8 and §2.8 or all of Chapter 4