

Contents

Preface	xiii
Glossary of Notation	xix
Picture Gallery	xxiii
Acknowledgments	xxv
I A Brief Introduction to Lattices	1
1 Basic Concepts	3
1.1 Ordering	3
1.1.1 Orders	3
1.1.2 Diagrams	5
1.1.3 Order constructions	5
1.1.4 Partitions	6
1.2 Lattices and semilattices	8
1.2.1 Lattices	8
1.2.2 Semilattices and closure systems	10
1.3 Some algebraic concepts	12
1.3.1 Homomorphisms	12
1.3.2 Sublattices	13
1.3.3 Congruences	14
2 Special Concepts	19
2.1 Elements and lattices	19
2.2 Direct and subdirect products	20
2.3 Polynomials and identities	23
2.4 Gluing	26

2.5	Modular and distributive lattices.....	30
2.5.1	The characterization theorems.....	30
2.5.2	Finite distributive lattices.....	31
2.5.3	Finite modular lattices.....	32
3	Congruences.....	35
3.1	Congruence spreading.....	35
3.2	Prime intervals.....	37
3.3	Congruence-preserving extensions and variants.....	39
II	Basic Techniques.....	45
4	Chopped Lattices.....	47
4.1	Basic definitions.....	47
4.2	Compatible vectors of elements.....	49
4.3	Compatible vectors of congruences.....	50
4.4	From the chopped lattice to the ideal lattice.....	52
4.5	Sectional complementation.....	53
5	Boolean Triples.....	57
5.1	The general construction.....	57
5.2	The congruence-preserving extension property.....	60
5.3	The distributive case.....	62
5.4	Two interesting intervals.....	63
6	Cubic Extensions.....	71
6.1	The construction.....	71
6.2	The basic property.....	73
III	Representation Theorems.....	77
7	The Dilworth Theorem.....	79
7.1	The representation theorem.....	79
7.2	<i>Proof-by-Picture</i>	80
7.3	Computing.....	82
7.4	Sectionally complemented lattices.....	83
7.5	Discussion.....	85
8	Minimal Representations.....	93
8.1	The results.....	93
8.2	<i>Proof-by-Picture</i> for minimal construction.....	94
8.3	The formal construction.....	95
8.4	<i>Proof-by-Picture</i> for minimality.....	97

8.5	Computing minimality	99
8.6	Discussion	100
9	Semimodular Lattices	105
9.1	The representation theorem	105
9.2	<i>Proof-by-Picture</i>	106
9.3	Construction and proof	107
9.4	Discussion	114
10	Modular Lattices	115
10.1	The representation theorem	115
10.2	<i>Proof-by-Picture</i>	116
10.3	Construction and proof	120
10.4	Discussion	125
11	Uniform Lattices	129
11.1	The representation theorem	129
11.2	<i>Proof-by-Picture</i>	129
11.3	The lattice $N(A, B)$	132
11.4	Formal proof	137
11.5	Discussion	139
IV	Extensions	143
12	Sectionally Complemented Lattices	145
12.1	The extension theorem	145
12.2	<i>Proof-by-Picture</i>	146
12.3	Simple extensions	148
12.4	Formal proof	150
12.5	Discussion	152
13	Semimodular Lattices	153
13.1	The extension theorem	153
13.2	<i>Proof-by-Picture</i>	153
13.3	The conduit	156
13.4	The construction	157
13.5	Formal proof	159
13.6	Discussion	159
14	Isoform Lattices	161
14.1	The result	161
14.2	<i>Proof-by-Picture</i>	161
14.3	Formal construction	165
14.4	The congruences	171

14.5	The isoform property	172
14.6	Discussion	173
15	Independence Theorems	177
15.1	Results	177
15.2	<i>Proof-by-Picture</i>	178
15.2.1	Frucht lattices	178
15.2.2	An automorphism-preserving simple extension	179
15.2.3	A congruence-preserving rigid extension	180
15.2.4	Merging the two extensions	181
15.2.5	The representation theorems	182
15.3	Formal proofs	183
15.3.1	An automorphism-preserving simple extension	183
15.3.2	A congruence-preserving rigid extension	185
15.3.3	Proof of the independence theorems	185
15.4	Discussion	187
16	Magic Wands	189
16.1	Constructing congruence lattices	189
16.1.1	Bijjective maps	189
16.1.2	Surjective maps	190
16.2	<i>Proof-by-Picture</i> for bijective maps	191
16.3	Verification for bijective maps	194
16.4	2/3-boolean triples	198
16.5	<i>Proof-by-Picture</i> for surjective maps	204
16.6	Verification for surjective maps	206
16.7	Discussion	207
V	Two Lattices	213
17	Sublattices	215
17.1	The results	215
17.2	<i>Proof-by-Picture</i>	217
17.3	Multi-coloring	219
17.4	Formal proof	220
17.5	Discussion	221
18	Ideals	227
18.1	The results	227
18.2	<i>Proof-by-Picture</i> for the main result	228
18.3	A very formal proof: Main result	230
18.3.1	Categoric preliminaries	230
18.3.2	From \mathbb{DI} to \mathbb{OR}	232

18.3.3 From OR to HE 232

18.3.4 From CH to DI 233

18.3.5 From HE to CH 234

18.3.6 From CH to LA 237

18.3.7 The final step 237

18.4 Proof for sectionally complemented lattices 238

18.5 *Proof-by-Picture* for planar lattices 241

18.6 Discussion 242

19 Tensor Extensions 245

19.1 The problem 245

19.2 Three unary functions 246

19.3 Defining tensor extensions 248

19.4 Computing 250

 19.4.1 Some special elements 250

 19.4.2 An embedding 252

 19.4.3 Distributive lattices 253

19.5 Congruences 254

 19.5.1 Congruence spreading 254

 19.5.2 Some structural observations 257

 19.5.3 Lifting congruences 259

 19.5.4 The main lemma 261

19.6 The congruence isomorphism 262

19.7 Discussion 263

Bibliography 265

Index 275