

# Contents

Editor's Note . . . . .	xi
Editor's Introduction . . . . .	xiii
<b>I Great Illusion of Twentieth Century Mathematics</b>	<b>21</b>
1 Theological Foundations	25
1.1 Potential and Actual Infinity . . . . .	25
1.1.1 Aurelius Augustinus (354–430) . . . . .	26
1.1.2 Thomas Aquinas (1225–1274) . . . . .	27
1.1.3 Giordano Bruno (1548–1600) . . . . .	29
1.1.4 Galileo Galilei (1564–1654) . . . . .	31
1.1.5 The Rejection of Actual Infinity . . . . .	33
1.1.6 Infinitesimal Calculus . . . . .	36
1.1.7 Number Magic . . . . .	37
1.1.8 Jean le Rond d'Alembert (1717–1783) . . . . .	39
1.2 The Disputation about Infinity in Baroque Prague . . . . .	41
1.2.1 Rodrigo de Arriaga (1592–1667) . . . . .	41
1.2.2 The Franciscan School . . . . .	47
1.3 Bernard Bolzano (1781–1848) . . . . .	48
1.3.1 Truth in Itself . . . . .	48
1.3.2 The Paradox of the Infinite . . . . .	52
1.3.3 Relational Structures on Infinite Multitudes . . . . .	54
1.4 Georg Cantor (1845–1918) . . . . .	56
1.4.1 Transfinite Ordinal Numbers . . . . .	56
1.4.2 Actual Infinity . . . . .	57
1.4.3 Rejection of Cantor's Theory . . . . .	58
2 Rise and Growth of Cantor's Set Theory	67
2.1 Basic Notions . . . . .	67
2.1.1 Relations and Functions . . . . .	70
2.1.2 Orderings . . . . .	72
2.1.3 Well-Orderings . . . . .	73
2.2 Ordinal Numbers . . . . .	76
2.3 Postulates of Cantor's Set Theory . . . . .	77
2.3.1 Cardinal Numbers . . . . .	79



2.3.2	Postulate of the Powerset . . . . .	81
2.3.3	Well-Ordering Postulate . . . . .	84
2.3.4	Objections of French Mathematicians . . . . .	86
2.4	Large Cardinalities . . . . .	89
2.4.1	Initial Ordinal Numbers . . . . .	89
2.4.2	Zorn's Lemma . . . . .	91
2.5	Developmental Influences . . . . .	92
2.5.1	Colonisation of Infinitary Mathematics . . . . .	92
2.5.2	Corpuses of Sets . . . . .	97
2.5.3	Introduction of Mathematical Formalism in Set Theory . . . . .	98
<b>3</b>	<b>Explication of the Problem</b>	<b>103</b>
3.1	Warnings . . . . .	103
3.2	Two Further Emphatic Warnings . . . . .	104
3.3	Ultrapower . . . . .	106
3.4	There Exists No Set of All Natural Numbers . . . . .	107
3.5	Unfortunate Consequences for All Infinitary Mathematics Based on Cantor's Set Theory . . . . .	109
<b>4</b>	<b>Summit and Fall</b>	<b>111</b>
4.1	Ultrafilters . . . . .	111
4.2	Basic Language of Set Theory . . . . .	113
4.3	Ultrapower Over a Covering Structure . . . . .	113
4.4	Ultraextension of the Domain of All Sets . . . . .	116
4.5	Ultraextension Operator . . . . .	118
4.6	Widening the Scope of Ultraextension Operator . . . . .	119
4.7	Non-existence of the Set of All Natural Numbers . . . . .	120
4.8	Extendable Domains of Sets . . . . .	121
4.9	The Problem of Infinity . . . . .	126
<b>II</b>	<b>New Theory of Sets and Semisets</b>	<b>129</b>
<b>5</b>	<b>Basic Notions</b>	<b>135</b>
5.1	Classes, Sets and Semisets . . . . .	135
5.2	Horizon . . . . .	136
5.3	Geometric Horizon . . . . .	141
5.4	Finite Natural Numbers . . . . .	143
<b>6</b>	<b>Extension of Finite Natural Numbers</b>	<b>145</b>
6.1	Natural Numbers within the Known Land of the Geometric Horizon . . . . .	145
6.2	Axiom of Prolongation . . . . .	147
6.3	Some Consequences of the Axiom of Prolongation . . . . .	148
6.4	Revealed Classes . . . . .	149



6.5	Forming Countable Classes . . . . .	152
6.6	Cuts on Natural Numbers . . . . .	157
7	Two Important Kinds of Classes . . . . .	159
7.1	Motivation – Primarily Evident Phenomena . . . . .	159
7.2	Mathematization: $\sigma$ -classes and $\pi$ -classes . . . . .	162
7.3	Applications . . . . .	165
7.4	Distortion of Natural Phenomena . . . . .	169
8	Hierarchy of Descriptive Classes . . . . .	171
8.1	Borel Classes . . . . .	171
8.2	Analytic Classes . . . . .	174
9	Topology . . . . .	177
9.1	Motivation – Medial Look at Sets . . . . .	177
9.2	Mathematization – Equivalence of Indiscernibility . . . . .	179
9.3	Historical Intermezzo . . . . .	183
9.4	The Nature of Topological Shapes . . . . .	184
9.5	Applications: Invisible Topological Shapes . . . . .	186
10	Synoptic Indiscernibility . . . . .	189
10.1	Synoptic Symmetry of Indiscernibility . . . . .	189
10.2	Geometric Equivalence of Indiscernibility . . . . .	192
11	Further Non-traditional Motivations . . . . .	197
11.1	Topological Misshapes . . . . .	197
11.2	Imaginary Semisets . . . . .	198
12	Search for Real Numbers . . . . .	201
12.1	Liberation of the Domain of Real Numbers . . . . .	201
12.2	Relation of Infinite Closeness on Rational Numbers in the Known Land of Geometric Horizon . . . . .	206
12.3	Real Numbers . . . . .	209
12.4	Intermezzo About the Stars in the Sky . . . . .	211
12.5	Interpretation of Real Numbers Corresponding to the First and Second Phase in Interpreting Stars in the Sky . . . . .	212
13	Classical Geometric World . . . . .	215
<b>III</b>	<b>Infinitesimal Calculus Reaffirmed</b> . . . . .	<b>217</b>
	Introduction . . . . .	219
14	Expansion of Ancient Geometric World . . . . .	225
14.1	Ancient and Classical Geometric Worlds . . . . .	225
14.2	Principles of Expansion . . . . .	226
14.3	Infinitely Large Natural Numbers . . . . .	227



14.4	Infinitely Large and Small Real Numbers . . . . .	228
14.5	Infinite Closeness . . . . .	230
14.6	Principles of Backward Projection . . . . .	231
14.7	Arithmetic with Improper Numbers $\infty, -\infty$ . . . . .	233
14.8	Further Fixed Notation for this Part . . . . .	235
15	Sequences of Numbers . . . . .	237
15.1	Binomial Numbers . . . . .	237
15.2	Limits of Sequences . . . . .	239
15.3	Euler's Number . . . . .	245
16	Continuity and Derivatives of Real Functions . . . . .	247
16.1	Continuity of a Function at a Point . . . . .	247
16.2	Derivative of a Function at a Point . . . . .	248
16.3	Functions Continuous on a Closed Interval . . . . .	251
16.4	Increasing and Decreasing Functions . . . . .	253
16.5	Continuous Bijective Functions . . . . .	254
16.6	Inverse Functions and Their Derivatives . . . . .	255
16.7	Higher-Order Derivatives, Extrema and Points of Inflection . . . . .	256
16.8	Limit of a Function at a Point . . . . .	259
16.9	Taylor's Expansion . . . . .	264
17	Elementary Functions and Their Derivatives . . . . .	267
17.1	Power Functions . . . . .	267
17.2	Exponential Function . . . . .	270
17.3	Logarithmic Function . . . . .	272
17.4	Derivatives of Power, Exponential and Logarithmic Functions . . . . .	274
17.5	Trigonometric Functions $\sin x, \cos x$ and Their Derivatives . . . . .	276
17.6	Trigonometric Functions $\tan x, \cot x$ and Their Derivatives . . . . .	281
17.7	Cyclometric Functions and Their Derivatives . . . . .	283
18	Numerical Series . . . . .	287
18.1	Convergence and Divergence . . . . .	287
18.2	Series with Non-negative Terms . . . . .	293
18.3	Convergence Criteria for Series with Positive Terms . . . . .	297
18.4	Absolutely and Non-absolutely Convergent Series . . . . .	300
19	Series of Functions . . . . .	305
19.1	Taylor and Maclaurin Series . . . . .	305
19.2	Maclaurin Series of the Exponential Function . . . . .	306
19.3	Maclaurin Series of Functions $\sin x, \cos x$ . . . . .	307
19.4	Powers of Complex Numbers . . . . .	308
19.5	Maclaurin Series of the Function $\log(1+x)$ for $-1 < x \leq 1$ . . . . .	310
19.6	Maclaurin Series of the Function $(1+x)^r$ for $ x  < 1$ . . . . .	312
19.7	Binomial Series $\sum \binom{r}{n} x^n$ for $x = \pm 1$ . . . . .	314
19.8	Series Expansion of the Function $\arctan x$ for $ x  \leq 1$ . . . . .	317



19.9 Uniform Convergence . . . . .	320
Appendix to Part III – Translation Rules	325
<b>IV Making Real Numbers Discrete</b>	<b>329</b>
Introduction . . . . .	331
20 Expansion of the Class Real of Real Numbers	333
20.1 Subsets of the Class Real . . . . .	333
20.2 Third Principle of Expansion . . . . .	334
21 Infinitesimal Arithmetics	337
21.1 Orders of Real Numbers . . . . .	337
21.2 Near-Equality . . . . .	338
22 Discretisation of the Ancient Geometric World	341
22.1 Grid . . . . .	341
22.2 Fourth Principle of Expansion . . . . .	343
22.3 Radius of Monads of a Full Almost-Uniform Grid . . . . .	344
Bibliography	347