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1.1 Some History

Let us start this historical introduction with a quote from the superb review article *On the Wonderful World of Random Walks* by E. W. Montroll and M. F. Shlesinger [116], which also contains a more detailed historical account of the development of probability theory:

Since traveling was onerous (and expensive), and eating, hunting and wenching generally did not fill the 17th century gentleman's day, two possibilities remained to occupy the empty hours, praying and gambling; many preferred the latter.

In fact, it is in the area of gambling that the theory of probability and stochastic processes has its origin. People had always engaged in gambling, but it was only through the thinking of the Enlightenment that the outcome of a gambling game was no longer seen as a divine decision, but became amenable to rational thinking and speculation. One of these 17th century gentlemen, a certain Chevalier de Méré, is reported to have posed a question concerning the odds at a gambling game to Pascal (1633–1662). The ensuing exchange of letters between Pascal and Fermat (1601–1665) on this problem is generally seen as the starting point of probability theory.

The first book on probability theory was written by Christian Huygens (1629–1695) in 1657 and had the title *De Ratiociniis in Ludo Aleae* (About the ratios in the game of dice). The first mathematical treatise on probability theory in the modern sense was Jacob Bernoulli's (1662–1705) book *Ars Conjectandi* (The art of making conjectures), which was published posthumously in 1713. It contained