Contents

1. A First Glimpse of Stochastic Processes 1
1.1 Some History 1
1.2 The Random Walk on a Line 3
1.2.1 From Binomial to Gaussian 6
1.2.2 From Binomial to Poisson 11
1.2.3 The Log-Normal Distribution 13
1.3 Further Reading 15

2. A Brief Survey of the Mathematics of Probability Theory 17
2.1 Some Basics of Probability Theory 17
2.1.1 Probability Spaces and Random Variables 18
2.1.2 Equivalent Measures 21
2.1.3 Distribution Functions and Probability Densities 22
2.1.4 Statistical Independence and Conditional Probabilities 23
2.1.5 The Central Limit Theorem 25
2.2 Stochastic Processes and Their Evolution Equations 27
2.2.1 Martingale Processes 30
2.2.2 Markov Processes 31
2.3 Itô Stochastic Calculus 40
2.3.1 Stochastic Integrals 40
2.3.2 Stochastic Differential Equations and the Itô Formula . 44
2.4 Summary 45
2.5 Further Reading 46

3. Diffusion Processes 49
3.1 The Random Walk Revisited 49
3.1.1 The Polya Problem 51
3.1.2 The Rayleigh–Pearson Walk 54
3.1.3 The Continuous-Time Random Walk 56
3.2 Free Brownian Motion 59
3.2.1 The Velocity Process 60
3.2.2 The Position Process 65
3.3 Brownian Motion in a Potential: The Kramers Problem 67

3.3.1 First Passage Time for One-Dimensional Fokker–Planck Equations 70
3.3.2 Kramers Result 72
3.4 Kinetic Ising Models and Monte Carlo Simulations 73
3.4.1 Probabilistic Structure 74
3.4.2 Monte Carlo Kinetics 74
3.4.3 The Mean-Field Kinetic Ising Model 76
3.5 Quantum Mechanics as a Diffusion Process 82
3.5.1 The Hydrodynamics of Brownian Motion 82
3.5.2 Conservative Diffusion Processes 85
3.5.3 The Hypothesis of Universal Brownian Motion 87
3.5.4 The Harmonic Oscillator and Quantum Fields 90
3.6 Summary 94
3.7 Further Reading 95

4. Beyond the Central Limit Theorem: Lévy Distributions 99
4.1 Back to Mathematics: Stable Distributions 100
4.2 The Weierstrass Random Walk 104
4.2.1 Definition and Solution 104
4.2.2 Superdiffusive Behavior 110
4.2.3 Generalization to Higher Dimensions 114
4.3 Fractal-Time Random Walks 117
4.3.1 A Fractal-Time Poisson Process 118
4.3.2 Subdiffusive Behavior 121
4.4 A Way to Avoid Diverging Variance: The Truncated Lévy Flight 122
4.5 Summary 126
4.6 Further Reading 128

5. Modeling the Financial Market 131
5.1 Basic Notions Pertaining to Financial Markets 132
5.2 Classical Option Pricing: The Black–Scholes Theory 141
5.2.1 The Black–Scholes Equation: Assumptions and Derivation 142
5.2.2 The Black–Scholes Equation: Solution and Interpretation 147
5.2.3 Risk-Neutral Valuation 152
5.2.4 Deviations from Black–Scholes: Implied Volatility 156
5.3 Models Beyond Geometric Brownian Motion 159
5.3.1 Statistical Analysis of Stock Prices 160
5.3.2 The Volatility Smile: Precursor to Gaussian Behavior? 172
5.4 Towards a Model of Financial Crashes 176
5.4.1 Some Empirical Properties 176
5.4.2 A Market Model: From Self-Organization to Criticality 180

5.5 Summary 188
5.6 Further Reading 190

A. Stable Distributions Revisited 191
A.1 Testing for Domains of Attraction 191
A.2 Closed-Form Expressions and Asymptotic Behavior 193

B. Hyperspherical Polar Coordinates 197

C. The Weierstrass Random Walk Revisited 201

D. The Exponentially Truncated Lévy Flight 207

E. Put–Call Parity 211

F. Geometric Brownian Motion 213

References 216

Index 225