Contents

	Preface	xiii
	Introduction	XXV
CHAPTER 1	Euclid's Geometry	1
	Very Brief Survey of the Beginnings of Geometry / 1 The Pythagoreans / 3 Plato / 5 Euclid of Alexandria / 7 The Axiomatic Method / 9 Undefined Terms / 11 Euclid's First Four Postulates / 15 The Parallel Postulate / 20 Attempts to Prove the Parallel Postulate / 23 The Danger in Diagrams / 25	
	The Power of Diagrams / 27 Straightedge-and-Compass Constructions, Briefly / 29 Descartes' Analytic Geometry and Broader Idea	of
	Constructions / 34 Briefly on the Number π / 38 Conclusion / 40	
CHAPTER 2	Logic and Incidence Geometry Elementary Logic / 53	53
	Theorems and Proofs / 55	

Dit / Bushavid

viii

RAA Proofs / 58

Negation / 60

Quantifiers / 61

Implication / 64

Law of Excluded Middle and Proof by Cases / 65

Brief Historical Remarks / 66

Incidence Geometry / 69

Models / 72

Consistency / 76

Isomorphism of Models / 79

Projective and Affine Planes / 81

Brief History of Real Projective Geometry / 89

Conclusion / 90

CHAPTER 3 Hilbert's Axioms

103

Flaws in Euclid / 103
Axioms of Betweenness / 105
Axioms of Congruence / 119
Axioms of Continuity / 129
Hilbert's Euclidean Axiom of Parallelism / 138
Conclusion / 142

CHAPTER 4 Neutral Geometry

161

Geometry Without a Parallel Axiom / 161
Alternate Interior Angle Theorem / 162
Exterior Angle Theorem / 164
Measure of Angles and Segments / 169
Equivalence of Euclidean Parallel Postulates / 173
Saccheri and Lambert Quadrilaterals / 176
Angle Sum of a Triangle / 183
Conclusion / 190

CHAPTER 5 History of the Parallel Postulate 209

Review / 209 Proclus / 210 Equidistance / 213
Wallis / 214
Saccheri / 218
Clairaut's Axiom and Proclus' Theorem / 219
Legendre / 221
Lambert and Taurinus / 223
Farkas Bolyai / 225

The Discovery of Non-Euclidean Geometry

János Bolyai / 239
Gauss / 242
Lobachevsky / 245
Subsequent Developments / 248
Non-Euclidean Hilbert Planes / 249
The Defect / 252
Similar Triangles / 253
Parallels Which Admit a Common
Perpendicular / 254
Limiting Parallel Rays, Hyperbolic Planes / 257
Classification of Parallels / 262
Strange New Universe? / 264

That The Parallel Postulate Postulate Parallel 289

detail Points in the Mignesbould Plane

Consistency of Hyperbolic Geometry / 289
Beltrami's Interpretation / 293
The Beltrami-Klein Model / 297
The Poincaré Models / 302
Perpendicularity in the Beltrami-Klein Model / 308
A Model of the Hyperbolic Plane
from Physics / 311
Inversion in Circles, Poincaré Congruence / 313
The Projective Nature of the Beltrami-Klein
Model / 333
Conclusion / 346

CHAPTER 8 Philosophical Implications, Fruitful Applications 371

What Is the Geometry of Physical Space? / 371
What Is Mathematics About? / 374
The Controversy about the Foundations of
Mathematics / 376
The Meaning / 380
The Fruitfulness of Hyperbolic Geometry for
Other Branches of Mathematics, Cosmology,
and Art / 382

CHAPTER 9 Geometric Transformations

Klein's Erlanger Programme / 397
Groups / 399
Applications to Geometric Problems / 403
Motions and Similarities / 408
Reflections / 411
Rotations / 414
Translations / 417
Half-Turns / 420
Ideal Points in the Hyperbolic Plane / 422
Parallel Displacements / 424
Glides / 426
Classification of Motions / 427
Automorphisms of the Cartesian Model / 431
Motions in the Poincaré Model / 436

CHAPTER 10 Further Results in Real Hyperbolic Geometry

Symmetry / 448

Congruence Described by Motions / 444

Area and Defect / 476
The Angle of Parallelism / 480
Cycles / 481

The Curvature of the Hyperbolic Plane / 483 Hyperbolic Trigonometry / 487 Circumference and Area of a Circle / 496 Saccheri and Lambert Quadrilaterals / 500 Coordinates in the Real Hyperbolic Plane / 507 The Circumscribed Cycle of a Triangle / 515 Bolyai's Constructions in the Hyperbolic Plane / 520

APPENDIX A Elliptic and Other Riemannian Geometries

APPENDIX B Hilbert's Geometry Without Real Numbers

e béen able la develop the theory

discusse that persists to this day.

der Geometrie en new conception emerged in which the existence of Axioms / 597

Bibliography / 603

Symbols / 611

Name Index / 613 dinds of students. Prospective high

nenoal and colleg Subject Index / 617

grometry (for example, the book was used very successing as part of

majors are given, in addition, detailed instruction in maniformation

geometry and hyperbolic trigonometry, challenging executions, and a his-