Contents

Preface to the Second Edition vii
Preface to the First Edition xi

I PROVABILITY

I Introduction to Formal Languages 3
1 General Information 3
2 First-Order Languages 5
Digression: Names 9
3 Beginners' Course in Translation 9
Digression: Syntax 15

II Truth and Deducibility 19
1 Unique Reading Lemma 19
2 Interpretation: Truth, Definability 23
3 Syntactic Properties of Truth 28
Digression: Natural Logic 33
4 Deducibility 36
Digression: Proof 45
5 Tautologies and Boolean Algebras 49
Digression: Kennings 53
6 Godel's Completeness Theorem 55
7 Countable Models and Skolem's Paradox 61
8 Language Extensions 66
9 Undefinability of Truth: The Language *SELF* 69
10 Smullyan's Language of Arithmetic 71
11 Undefinability of Truth: Tarski's Theorem 74
Digression: Self-Reference 77
12 Quantum Logic 78
Appendix: The Von Neumann Universe 89
The Last Digression. Truth as Value and Duty: Lessons of Mathematics 96

III The Continuum Problem and Forcing 105
1 The Problem: Results, Ideas 105
2 A Language of Real Analysis 110
3 The Continuum Hypothesis Is Not Deducible in L_2 Real 114

4 Boolean-Valued Universes 120
5 The Axiom of Extensionality Is "True" 124
6 The Axioms of Pairing, Union, Power Set, and Regularity Are "True" 127
7 The Axioms of Infinity, Replacement, and Choice Are "True" 132
8 The Continuum Hypothesis Is "False" for Suitable B 140
9 Forcing 145

IV The Continuum Problem and Constructible Sets 151
1 Gödel's Constructible Universe 151
2 Definability and Absoluteness 155
3 The Constructible Universe as a Model for Set Theory 158
4 The Generalized Continuum Hypothesis Is L-True 161
5 Constructibility Formula 164
6 Remarks on Formalization 171
7 What Is the Cardinality of the Continuum? 172

II COMPUTABILITY

V Recursive Functions and Church's Thesis 179
1 Introduction. Intuitive Computability 179
2 Partial Recursive Functions 183
3 Basic Examples of Recursiveness 187
4 Enumerable and Decidable Sets 191
5 Elements of Recursive Geometry 201

VI Diophantine Sets and Algorithmic Undecidability 207
1 The Basic Result 207
2 Plan of Proof 209
3 Enumerable Sets Are D-Sets 211
4 The Reduction 214
5 Construction of a Special Diophantine Set 217
6 The Graph of the Exponential Is Diophantine 221
7 The Factorial and Binomial Coefficient Graphs Are Diophantine 221
8 Versal Families 223
9 Kolmogorov Complexity 226

III PROVABILITY AND COMPUTABILITY

VII Gödel's Incompleteness Theorem 235
1 Arithmetic of Syntax 235
2 Incompleteness Principles 240
3 Nonenumerability of True Formulas 241

4 Syntactic Analysis 243
5 Enumerability of Deducible Formulas 249
6 The Arithmetical Hierarchy 252
7 Productivity of Arithmetical Truth 255
8 On the Length of Proofs 258

VIII Recursive Groups 263
1 Basic Result and Its Corollaries 263
2 Free Products and HNN-Extensions 266
3 Embeddings in Groups with Two Generators 270
4 Benign Subgroups 271
5 Bounded Systems of Generators 275
6 End of the Proof 280

IX Constructive Universe and Computation 285
1 Introduction: A Categorical View of Computation 285
2 Expanding Constructive Universe: Generalities 289
3 Expanding Constructive Universe: Morphisms 293
4 Operads and PROPs 296
5 The World of Graphs as a Topological Language 298
6 Models of Computation and Complexity 307
7 Basics of Quantum Computation I: Quantum Entanglement . . 315
8 Selected Quantum Subroutines 319
9 Shor's Factoring Algorithm 322
10 Kolmogorov Complexity and Growth of Recursive Functions . . 325

IV MODEL THEORY

X Model Theory 331
1 Languages and Structures 331
2 The Compactness Theorem 334
3 Basic Methods and Constructions 342
4 Completeness and Quantifier Elimination in Some Theories . . . 350
5 Classification Theory 359
6 Geometric Stability Theory 364
7 Other Languages and Nonelementary Model Theory 374

Suggestions for Further Reading 379

Index 381