

# Table of Contents

Chapter 0. Introduction . . . . .	1
A. Brief Definitions and Motivation . . . . .	1
B. Why Write a Book on Einstein Manifolds? . . . . .	5
C. Existence . . . . .	6
D. Examples . . . . .	
1. Algebraic Examples . . . . .	6
2. Examples from Analysis . . . . .	7
3. Sporadic Examples . . . . .	8
E. Uniqueness and Moduli . . . . .	9
F. A Brief Survey of Chapter Contents . . . . .	10
G. Leitfaden . . . . .	14
H. Getting the Feel of Ricci Curvature . . . . .	15
I. The Main Problems Today . . . . .	18
Chapter 1. Basic Material . . . . .	20
A. Introduction . . . . .	20
B. Linear Connections . . . . .	22
C. Riemannian and Pseudo-Riemannian Manifolds . . . . .	29
D. Riemannian Manifolds as Metric Spaces . . . . .	35
E. Riemannian Immersions, Isometries and Killing Vector Fields . . . . .	37
F. Einstein Manifolds . . . . .	41
G. Irreducible Decompositions of Algebraic Curvature Tensors . . . . .	45
H. Applications to Riemannian Geometry . . . . .	48
I. Laplacians and Weitzenböck Formulas . . . . .	52
J. Conformal Changes of Riemannian Metrics . . . . .	58
K. First Variations of Curvature Tensor Fields . . . . .	62
Chapter 2. Basic Material (Continued): Kähler Manifolds . . . . .	66
0. Introduction . . . . .	66
A. Almost Complex and Complex Manifolds . . . . .	66
B. Hermitian and Kähler Metrics . . . . .	69
C. Ricci Tensor and Ricci Form . . . . .	73
D. Holomorphic Sectional Curvature . . . . .	75
E. Chern Classes . . . . .	78



F. The Ricci Form as the Curvature Form of a Line Bundle.....	81
G. Hodge Theory.....	83
H. Holomorphic Vector Fields and Infinitesimal Isometries.....	86
I. The Calabi-Futaki Theorem.....	92
 Chapter 3. Relativity.....	 94
A. Introduction.....	94
B. Physical Interpretations.....	94
C. The Einstein Field Equation.....	96
D. Tidal Stresses.....	97
E. Normal Forms for Curvature.....	98
F. The Schwarzschild Metric.....	101
G. Planetary Orbits.....	105
H. Perihelion Precession.....	107
I. Geodesics in the Schwarzschild Universe.....	108
J. Bending of Light.....	110
K. The Kruskal Extension.....	111
L. How Completeness May Fail.....	113
M. Singularity Theorems.....	115
 Chapter 4. Riemannian Functionals.....	 116
A. Introduction.....	116
B. Basic Properties of Riemannian Functionals.....	117
C. The Total Scalar Curvature: First Order Properties.....	119
D. Existence of Metrics with Constant Scalar Curvature.....	122
E. The Image of the Scalar Curvature Map.....	124
F. The Manifold of Metrics with Constant Scalar Curvature.....	126
G. Back to the Total Scalar Curvature: Second Order Properties.....	129
H. Quadratic Functionals.....	133
 Chapter 5. Ricci Curvature as a Partial Differential Equation.....	 137
A. Pointwise (Infinitesimal) Solvability.....	137
B. From Pointwise to Local Solvability: Obstructions.....	138
C. Local Solvability of $\text{Ric}(g) = r$ for Nonsingular $r$ .....	140
D. Local Construction of Einstein Metrics.....	142
E. Regularity of Metrics with Smooth Ricci Tensors.....	143
F. Analyticity of Einstein Metrics and Applications.....	145
G. Einstein Metrics on Three-Manifolds.....	146
H. A Uniqueness Theorem for Ricci Curvature.....	152
I. Global Non-Existence.....	153
 Chapter 6. Einstein Manifolds and Topology.....	 154
A. Introduction.....	154
B. Existence of Einstein Metrics in Dimension 2.....	155
C. The 3-Dimensional Case.....	157



D. The 4-Dimensional Case .....	161
E. Ricci Curvature and the Fundamental Group .....	165
F. Scalar Curvature and the Spinorial Obstruction .....	169
G. A Proof of the Cheeger-Gromoll Theorem on Complete Manifolds with Non-Negative Ricci Curvature .....	171
Chapter 7. Homogeneous Riemannian Manifolds .....	177
A. Introduction .....	177
B. Homogeneous Riemannian Manifolds .....	178
C. Curvature .....	181
D. Some Examples of Homogeneous Einstein Manifolds .....	186
E. General Results on Homogeneous Einstein Manifolds .....	189
F. Symmetric Spaces .....	191
G. Standard Homogeneous Riemannian Manifolds .....	196
H. Tables .....	200
I. Remarks on Homogeneous Lorentz Manifolds .....	205
Chapter 8. Compact Homogeneous Kähler Manifolds .....	208
0. Introduction .....	208
A. The Orbits of a Compact Lie Group for the Adjoint Representation ..	209
B. The Canonical Complex Structure .....	212
C. The $G$ -Invariant Ricci Form .....	215
D. The Symplectic Structure of Kirillov-Kostant-Souriau .....	220
E. The Invariant Kähler Metrics on the Orbits .....	221
F. Compact Homogeneous Kähler Manifolds .....	224
G. The Space of Orbits .....	227
H. Examples .....	229
Chapter 9. Riemannian Submersions .....	235
A. Introduction .....	235
B. Riemannian Submersions .....	236
C. The Invariants $A$ and $T$ .....	238
D. O'Neill's Formulas for Curvature .....	241
E. Completeness and Connections .....	244
F. Riemannian Submersions with Totally Geodesic Fibres .....	249
G. The Canonical Variation .....	252
H. Applications to Homogeneous Einstein Manifolds .....	256
I. Further Examples of Homogeneous Einstein Manifolds .....	263
J. Warped Products .....	265
K. Examples of Non-Homogeneous Compact Einstein Manifolds with Positive Scalar Curvature .....	272
Chapter 10. Holonomy Groups .....	278
A. Introduction .....	278
B. Definitions .....	280



C. Covariant Derivative Vanishing Versus Holonomy Invariance. Examples.....	282
D. Riemannian Products Versus Holonomy .....	285
E. Structure I.....	288
F. Holonomy and Curvature.....	290
G. Symmetric Spaces; Their Holonomy .....	294
H. Structure II .....	300
I. The Non-Simply Connected Case .....	307
J. Lorentzian Manifolds .....	309
K. Tables .....	311
 Chapter 11. Kähler-Einstein Metrics and the Calabi Conjecture.....	 318
A. Kähler-Einstein Metrics .....	318
B. The Resolution of the Calabi Conjecture and its Consequences.....	322
C. A Brief Outline of the Proofs of the Aubin-Calabi-Yau Theorems.....	326
D. Compact Complex Manifolds with Positive First Chern Class .....	329
E. Extremal Metrics .....	333
 Chapter 12. The Moduli Space of Einstein Structures.....	 340
A. Introduction .....	340
B. Typical Examples: Surfaces and Flat Manifolds.....	342
C. Basic Tools .....	345
D. Infinitesimal Einstein Deformations.....	346
E. Formal Integrability .....	348
F. Structure of the Premoduli Spaces.....	351
G. The Set of Einstein Constants.....	352
H. Rigidity of Einstein Structures .....	355
I. Dimension of the Moduli Space.....	358
J. Deformations of Kähler-Einstein Metrics.....	361
K. The Moduli Space of the Underlying Manifold of K3 Surfaces .....	365
 Chapter 13. Self-Duality .....	 369
A. Introduction .....	369
B. Self-Duality.....	370
C. Half-Conformally Flat Manifolds .....	372
D. The Penrose Construction.....	379
E. The Reverse Penrose Construction .....	385
F. Application to the Construction of Half-Conformally Flat Einstein Manifolds .....	390
 Chapter 14. Quaternion-Kähler Manifolds.....	 396
A. Introduction .....	396
B. Hyperkählerian Manifolds .....	398
C. Examples of Hyperkählerian Manifolds .....	400



D. Quaternion-Kähler Manifolds .....	402
E. Symmetric Quaternion-Kähler Manifolds.....	408
F. Quaternionic Manifolds .....	410
G. The Twistor Space of a Quaternionic Manifold .....	412
H. Applications of the Twistor Space Theory .....	415
I. Examples of Non-Symmetric Quaternion-Kähler Manifolds .....	419
Chapter 15. A Report on the Non-Compact Case .....	422
A. Introduction .....	422
B. A Construction of Nonhomogeneous Einstein Metrics .....	423
C. Bundle Constructions .....	424
D. Bounded Domains of Holomorphy .....	428
Chapter 16. Generalizations of the Einstein Condition .....	432
A. Introduction .....	432
B. Natural Linear Conditions on $Dr$ .....	433
C. Codazzi Tensors.....	436
D. The Case $Dr \in C^\infty(Q \oplus S)$ : Riemannian Manifolds with Harmonic Weyl Tensor .....	440
E. Condition $Dr \in C^\infty(S)$ : Riemannian Manifolds with Harmonic Curvature .....	443
F. The Case $Dr \in C^\infty(Q)$ .....	447
G. Condition $Dr \in C^\infty(A)$ : Riemannian Manifolds such that $(D_X r)(X, X) = 0$ for all Tangent Vectors $X$ .....	450
H. Oriented Riemannian 4-Manifolds with $\delta W^+ = 0$ .....	451
Appendix. Sobolev Spaces and Elliptic Operators .....	456
A. Hölder Spaces.....	456
B. Sobolev Spaces.....	457
C. Embedding Theorems .....	457
D. Differential Operators .....	459
E. Adjoint .....	460
F. Principal Symbol .....	460
G. Elliptic Operators.....	461
H. Schauder and $L^p$ Estimates for Linear Elliptic Operators.....	463
I. Existence for Linear Elliptic Equations.....	464
J. Regularity of Solutions for Elliptic Equations .....	466
K. Existence for Nonlinear Elliptic Equations.....	467
Addendum .....	471
A. Infinitely Many Einstein Constants on $S^2 \times S^{2m+1}$ .....	471
B. Explicit Metrics with Holonomy $G_2$ and $\text{Spin}(7)$ .....	472
C. Inhomogeneous Kähler-Einstein Metrics with Positive Scalar Curvature .....	474



D. Uniqueness of Kähler-Einstein Metrics with Positive Scalar Curvature ..... 475

E. Hyperkählerian Quotients..... 477

Bibliography ..... 479

Notation Index ..... 500

Subject Index..... 505