

# Contents

<b>1</b>	<b>Pre-Hellenic Antiquity</b> . . . . .	1
1.1	Prehistory . . . . .	1
1.2	Egypt . . . . .	3
1.3	Mesopotamia . . . . .	5
1.4	Problems . . . . .	6
1.5	Exercises . . . . .	7
<b>2</b>	<b>Some Pioneers of Greek Geometry</b> . . . . .	9
2.1	Thales of Miletus . . . . .	10
2.2	Pythagoras and the Golden Ratio . . . . .	13
2.3	Trisecting the Angle . . . . .	16
2.4	Squaring the Circle . . . . .	18
2.5	Duplicating the Cube . . . . .	23
2.6	Incommensurable Magnitudes . . . . .	29
2.7	The Method of Exhaustion . . . . .	34
2.8	On the Continuity of Space . . . . .	38
2.9	Problems . . . . .	40
2.10	Exercises . . . . .	41
<b>3</b>	<b>Euclid's Elements</b> . . . . .	43
3.1	Book 1: Straight Lines . . . . .	44
3.2	Book 2: Geometric Algebra . . . . .	64
3.3	Book 3: Circles . . . . .	68
3.4	Book 4: Polygons . . . . .	74
3.5	Book 5: Ratios . . . . .	77
3.6	Book 6: Similarities . . . . .	78
3.7	Book 7: Divisibility in Arithmetic . . . . .	85
3.8	Book 8: Geometric Progressions . . . . .	90
3.9	Book 9: More on Numbers . . . . .	90
3.10	Book 10: Incommensurable Magnitudes . . . . .	91
3.11	Book 11: Solid Geometry . . . . .	92
3.12	Book 12: The Method of Exhaustion . . . . .	100

3.13	Book 13: Regular Polyhedrons . . . . .	106
3.14	Problems . . . . .	109
3.15	Exercises . . . . .	110
<b>4</b>	<b>Some Masters of Greek Geometry . . . . .</b>	<b>111</b>
4.1	Archimedes on the Circle . . . . .	112
4.2	Archimedes on the Number $\pi$ . . . . .	113
4.3	Archimedes on the Sphere . . . . .	120
4.4	Archimedes on the Parabola . . . . .	124
4.5	Archimedes on the Spiral . . . . .	127
4.6	Apollonius on Conical Sections . . . . .	130
4.7	Apollonius on Conjugate Directions . . . . .	135
4.8	Apollonius on Tangents . . . . .	139
4.9	Apollonius on Poles and Polar Lines . . . . .	143
4.10	Apollonius on Foci . . . . .	146
4.11	Heron on the Triangle . . . . .	149
4.12	Menelaus on Trigonometry . . . . .	151
4.13	Ptolemy on Trigonometry . . . . .	154
4.14	Pappus on Anharmonic Ratios . . . . .	157
4.15	Problems . . . . .	162
4.16	Exercises . . . . .	164
<b>5</b>	<b>Post-Hellenic Euclidean Geometry . . . . .</b>	<b>167</b>
5.1	Still Chasing the Number $\pi$ . . . . .	168
5.2	The Medians of a Triangle . . . . .	170
5.3	The Altitudes of a Triangle . . . . .	172
5.4	Ceva's Theorem . . . . .	172
5.5	The Trisectrices of a Triangle . . . . .	174
5.6	Another Look at the Foci of Conics . . . . .	177
5.7	Inversions in the Plane . . . . .	180
5.8	Inversions in Solid Space . . . . .	184
5.9	The Stereographic Projection . . . . .	186
5.10	Let us Burn our Rulers! . . . . .	189
5.11	Problems . . . . .	195
5.12	Exercises . . . . .	195
<b>6</b>	<b>Projective Geometry . . . . .</b>	<b>197</b>
6.1	Perspective Representation . . . . .	198
6.2	Projective Versus Euclidean . . . . .	202
6.3	Anharmonic Ratio . . . . .	205
6.4	The Desargues and the Pappus Theorems . . . . .	208
6.5	Axiomatic Projective Geometry . . . . .	210
6.6	Arguesian and Pappian Planes . . . . .	214
6.7	The Projective Plane over a Skew Field . . . . .	219
6.8	The Hilbert Theorems . . . . .	222
6.9	Problems . . . . .	240
6.10	Exercises . . . . .	241

<b>7</b>	<b>Non-Euclidean Geometry</b> . . . . .	243
7.1	Chasing Euclid's Fifth Postulate . . . . .	245
7.2	The Saccheri Quadrilaterals . . . . .	251
7.3	The Angles of a Triangle . . . . .	259
7.4	The Limit Parallels . . . . .	264
7.5	The Area of a Triangle . . . . .	273
7.6	The Beltrami–Klein and Poincaré Disks . . . . .	280
7.7	Problems . . . . .	302
7.8	Exercises . . . . .	303
<b>8</b>	<b>Hilbert's Axiomatization of the Plane</b> . . . . .	305
8.1	The Axioms of Incidence . . . . .	306
8.2	The Axioms of Order . . . . .	307
8.3	The Axioms of Congruence . . . . .	319
8.4	The Axiom of Continuity . . . . .	335
8.5	The Axioms of Parallelism . . . . .	351
8.6	Problems . . . . .	353
8.7	Exercises . . . . .	353
<b>Appendix A Constructibility</b> . . . . .		355
A.1	The Minimal Polynomial . . . . .	355
A.2	The Eisenstein Criterion . . . . .	358
A.3	Ruler and Compass Constructibility . . . . .	360
A.4	Constructibility <i>Versus</i> Field Theory . . . . .	363
<b>Appendix B The Classical Problems</b> . . . . .		369
B.1	Duplicating the Cube . . . . .	369
B.2	Trisecting the Angle . . . . .	369
B.3	Squaring the Circle . . . . .	371
<b>Appendix C Regular Polygons</b> . . . . .		379
C.1	What the Greek Geometers Knew . . . . .	379
C.2	The Problem in Algebraic Terms . . . . .	380
C.3	Fermat Primes . . . . .	382
C.4	Elements of Modular Arithmetic . . . . .	384
C.5	A Flavour of Galois Theory . . . . .	387
C.6	The Gauss–Wantzel Theorem . . . . .	390
<b>References and Further Reading</b> . . . . .		395
<b>Index</b> . . . . .		397